Fundamentals of Electromagnetics

Electric Field, Potential, and Capacitance

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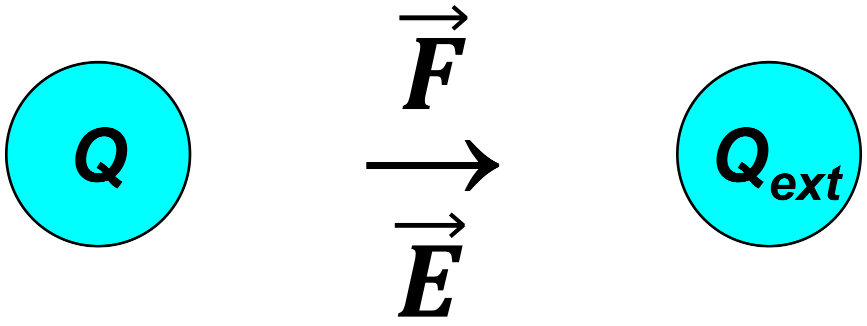
Topics

- **Electric Field and Force**
- **Potential and Capacitance**
- **Permittivity**
- **Displacement Current**
- **(Virtual) Demonstration: Capacitive Coupling**
- **Gauss's Law: Electric Field, Potential, and Capacitance**

- ***Material taken from "Fundamentals of Electromagnetics" video series***
  - *Publicly available on YouTube; search for above title*
  - *Direct playlist link:*
    - https://youtube.com/playlist?list=PLtrpQ-gPvnJn2r9Mw49jij7Ky0mb6RJYF&si=UxEKqVRgsR9w6nZ7

Electric Field, Force, and Potential



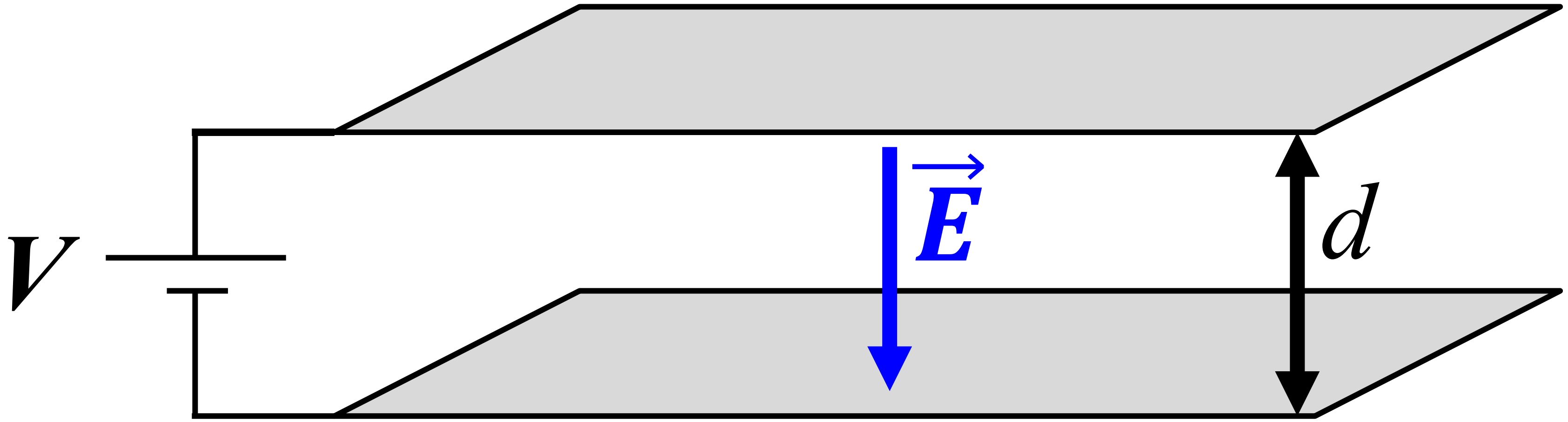
$$\vec{E} = \frac{\vec{F}}{Q_{ext}}$$

$$\vec{E}(units) = \frac{Newton}{Coulomb} = \frac{N}{C}$$

$$= \frac{N \cdot m}{C \cdot m} = \frac{J}{C \cdot m}$$

$$= \frac{V}{m}$$

Electric Field and Potential: Parallel Plates Example



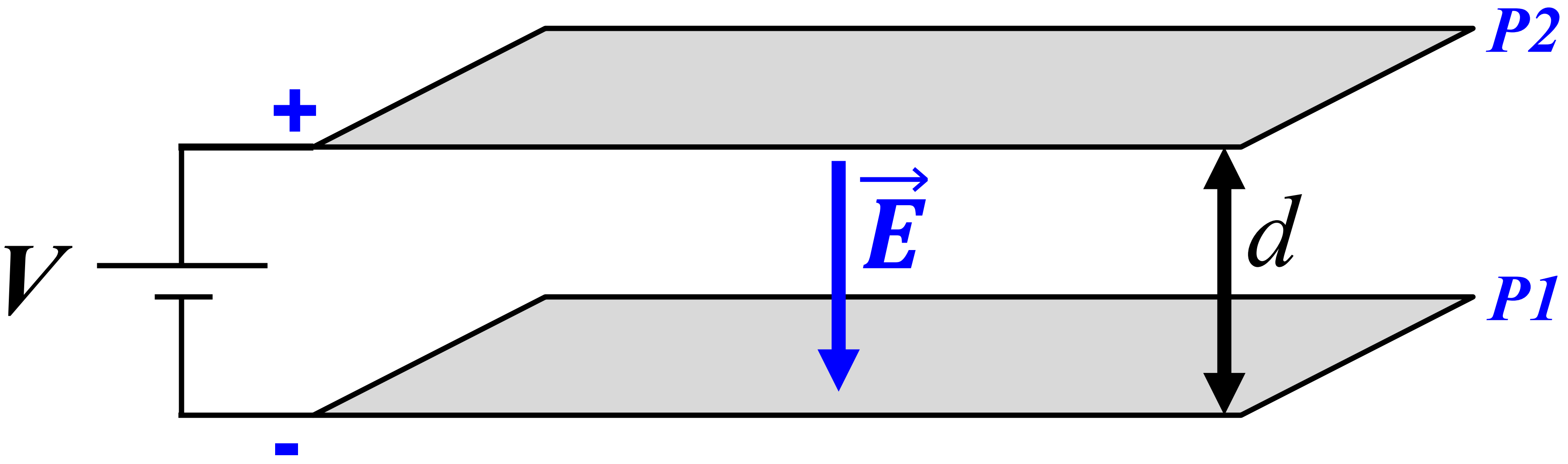
$$E = \frac{V}{d}$$

$$\frac{1\,V}{1\,m} = 1\,V/m$$

$$\frac{10\,mV}{1\,cm} = 1\,V/m$$

$$\frac{1\,mV}{1\,mm} = 1\,V/m$$

Electric Field and Potential (cont.)

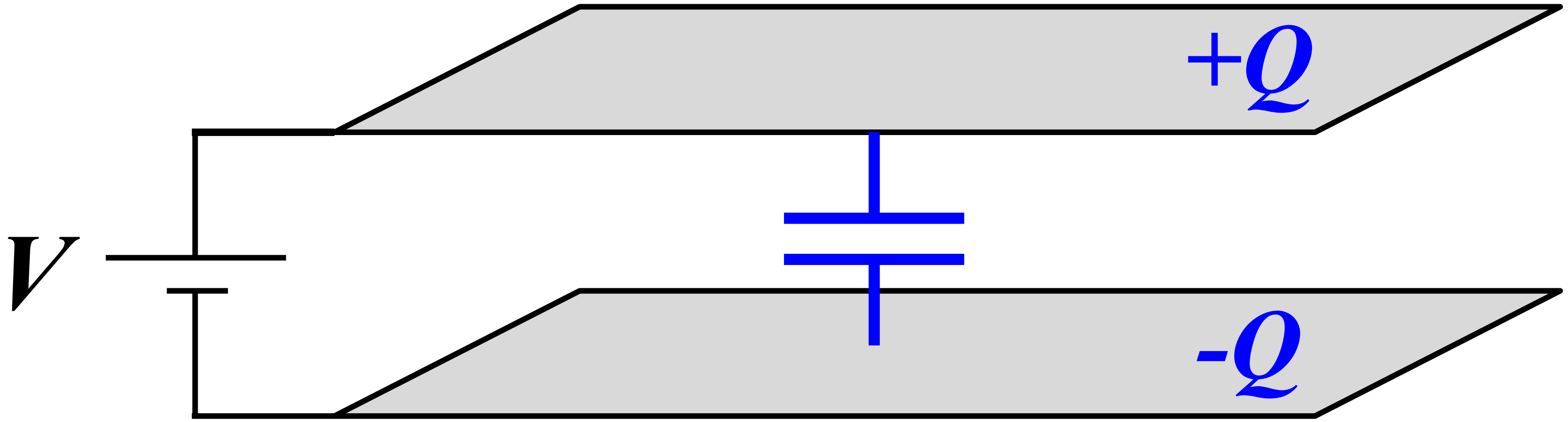


$$\vec{\boldsymbol{E}} = -\nabla V = -\left(\frac{dV}{dx}\hat{\boldsymbol{x}} + \frac{dV}{dy}\hat{\boldsymbol{y}} + \frac{dV}{dz}\hat{\boldsymbol{z}}\right)$$

$$V = -\int_{P1}^{P2} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{l}}$$

Electric field vector points in opposite direction to applied potential

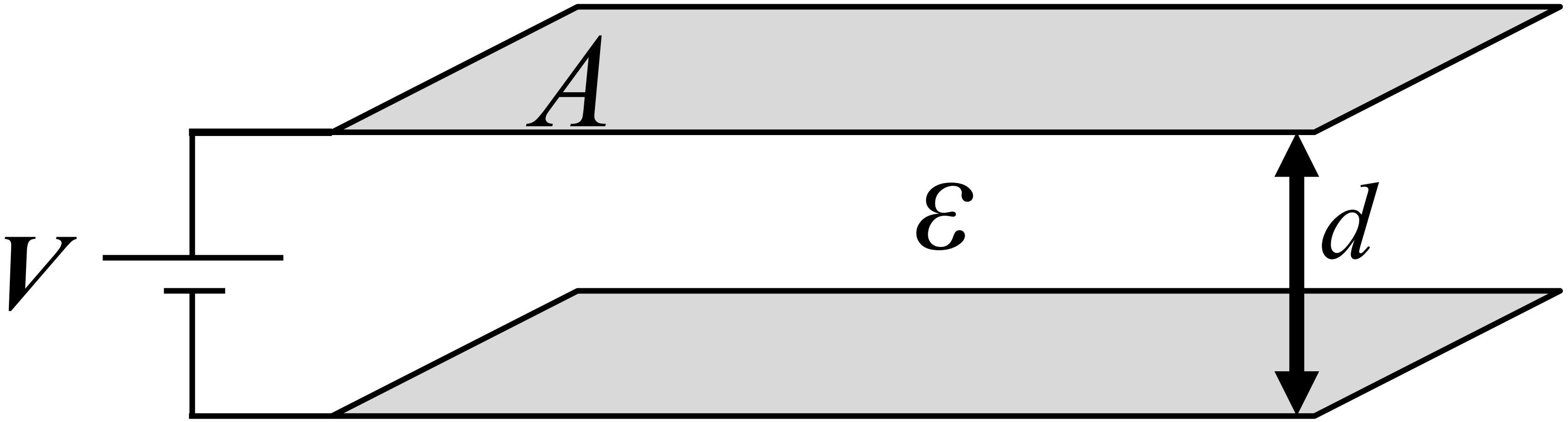
Capacitance



$$C = \frac{Q}{V} \qquad \frac{Coulombs}{Volts}$$

$$= Farads$$

Capacitance (cont.)



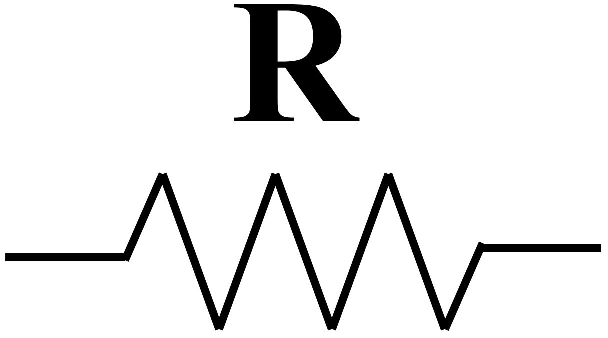
$$C = \frac{\varepsilon A}{d}$$

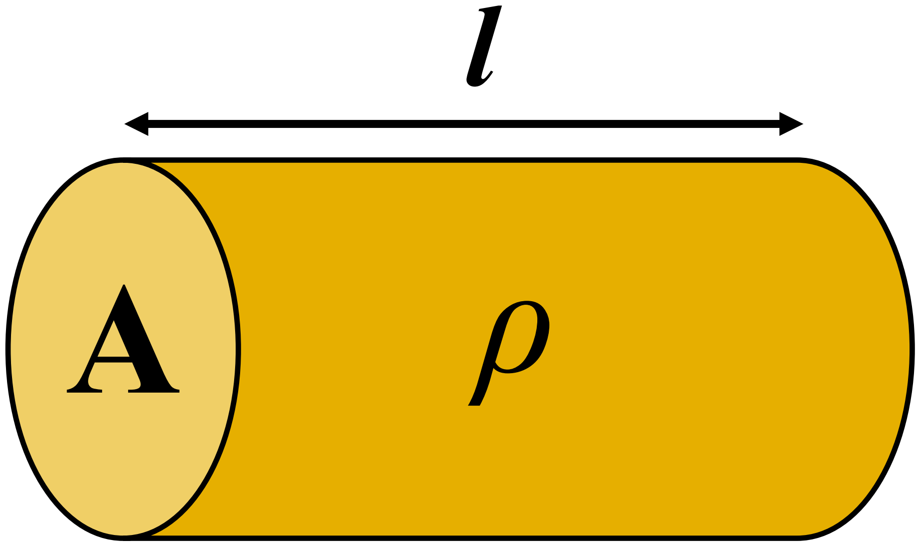
ε

Permittivity

???

Resistance, Resistivity, and Conductivity





$$R = \frac{\rho l}{A}$$

$$\Omega = \frac{(\rho) \cdot meters}{meters^2}$$

$$\rho(units) = \Omega \cdot m$$

$$\sigma = \frac{1}{\rho} \qquad \sigma(units) = \frac{\Omega^{-1}}{m} \qquad (\mho)$$

Permittivity (revisited)

$$\varepsilon = \varepsilon_r \varepsilon_0$$

$$\varepsilon_r = relative\ permittivity$$

$$\varepsilon_0 = permittivity\ of\ free\ space$$

Permittivity of Free Space (Vacuum)

$$\varepsilon = \varepsilon_r \varepsilon_0$$

$$\varepsilon_0 = permittivity\ of\ free\ space$$

$$\varepsilon_0 \approx \frac{1}{36\pi} \times 10^{-9}\ \frac{F}{m}$$

$$\varepsilon_0 \approx 8.84\ pF/m$$

Relative Permittivity (Dielectric Constant)

$$\varepsilon = \varepsilon_r \varepsilon_0$$

$$\varepsilon_r = relative\ permittivity$$

$$= dielectric\ constant$$

$$\varepsilon_r\,(free\ space/vacuum) = 1$$

$$\varepsilon_r\,(air) = 1.00056 \approx 1$$

$$\varepsilon = \varepsilon_r \varepsilon_0$$

$$\varepsilon_r = relative\ permittivity$$

$$= dielectric\ constant$$

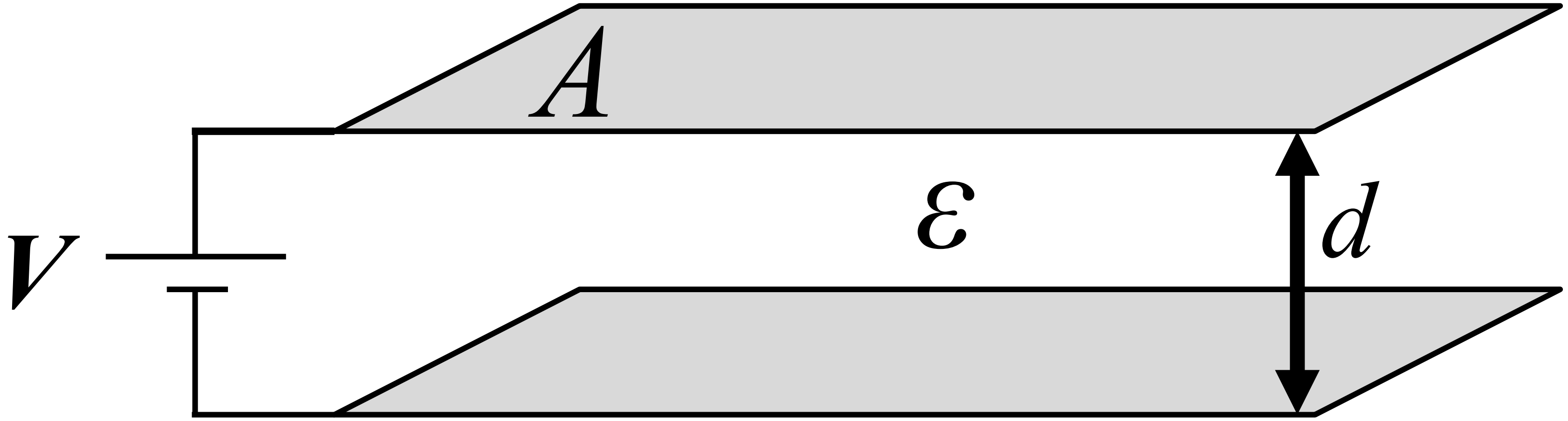
$$\varepsilon_r\,(ceramics) \approx 5\text{-}10$$

$$\varepsilon_r\,(Teflon) \approx 2$$

$$\varepsilon_r\,(tantalum\ oxide) \approx 27$$

$$\varepsilon_r\,(G10) \approx 5$$

Permittivity and Capacitance



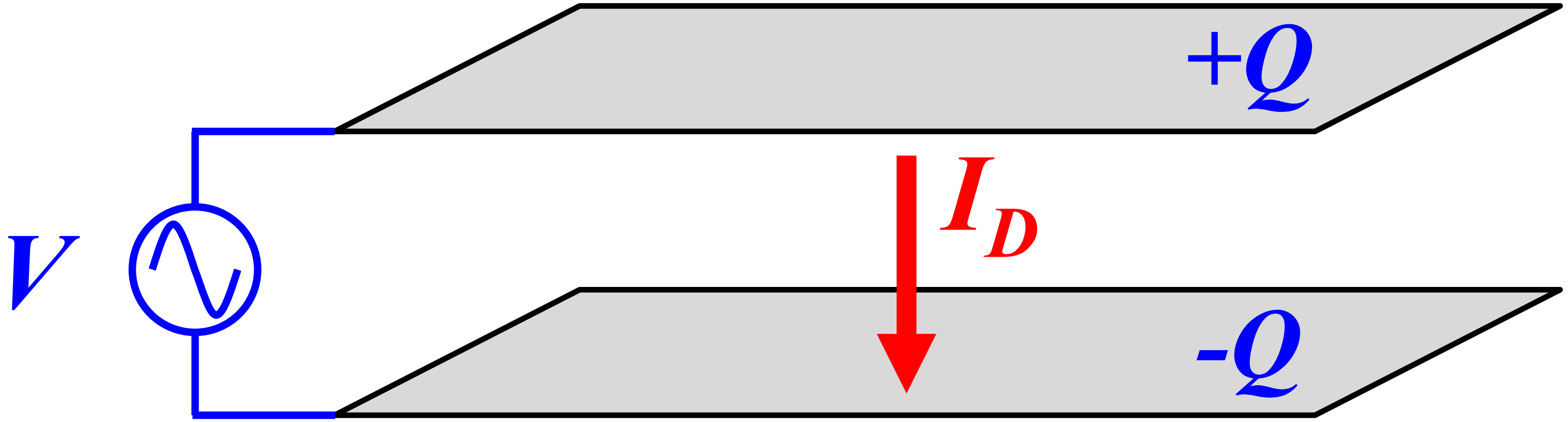
$$C = \frac{\varepsilon A}{d} \qquad \frac{\left(\frac{Farad}{m}\right) \cdot m^2}{m}$$

$$= \boldsymbol{Farads}$$

"Capacitivity"

Units of "Farads/meter" only meaningful when applied to specific geometry

Time-Varying Potential and Displacement Current

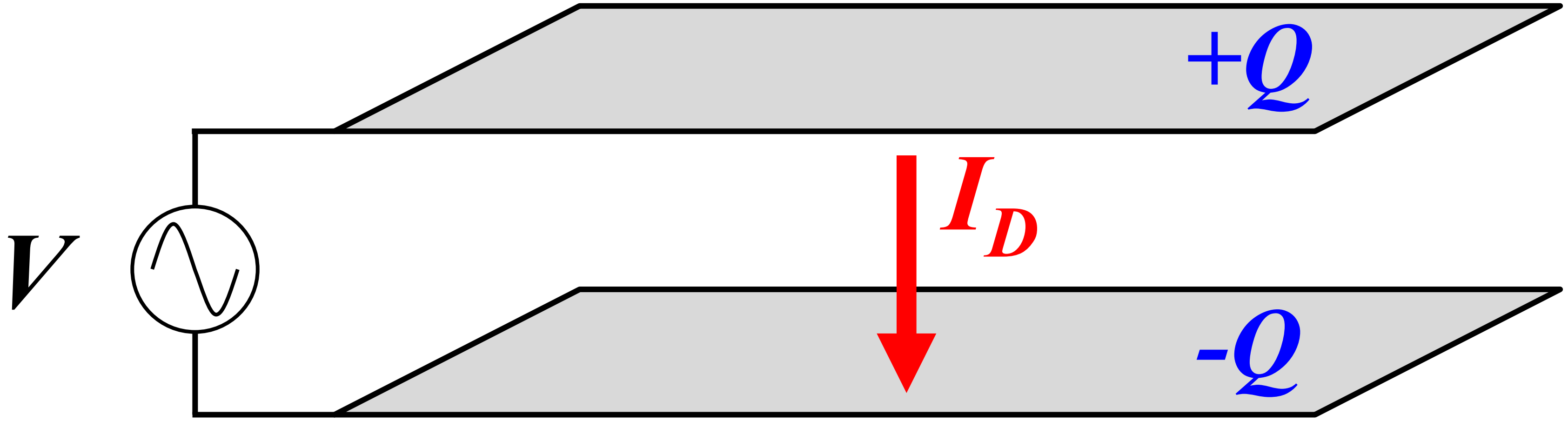


$$C = \frac{Q}{V} \qquad CV = Q$$

$$C\frac{dV}{dt} = \frac{dQ}{dt} = I_D$$

Displacement current

Capacitive Reactance (Impedance)



$$V(t) = Ve^{j\omega t}$$

$$\frac{dV(t)}{dt} = j\omega V e^{j\omega t}$$

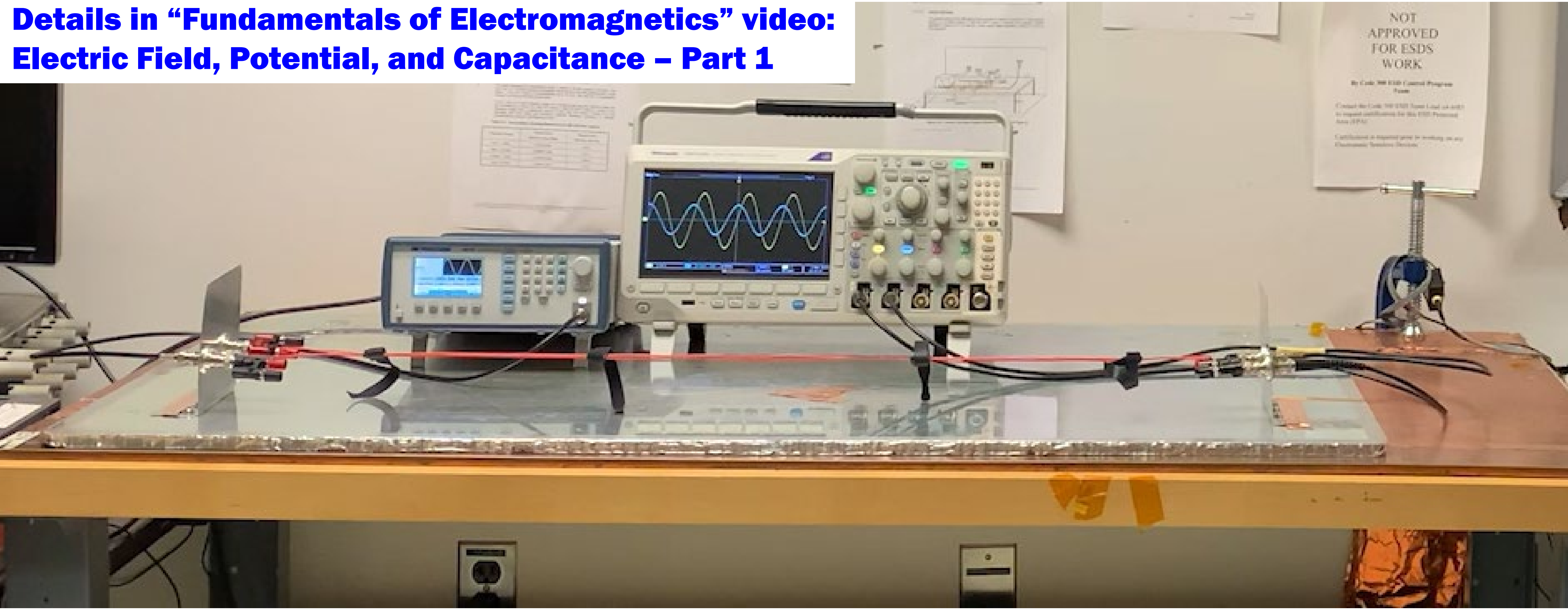
$$C\frac{dV}{dt} = \frac{dQ}{dt} = I_D$$

$$j\omega VC = I_D$$

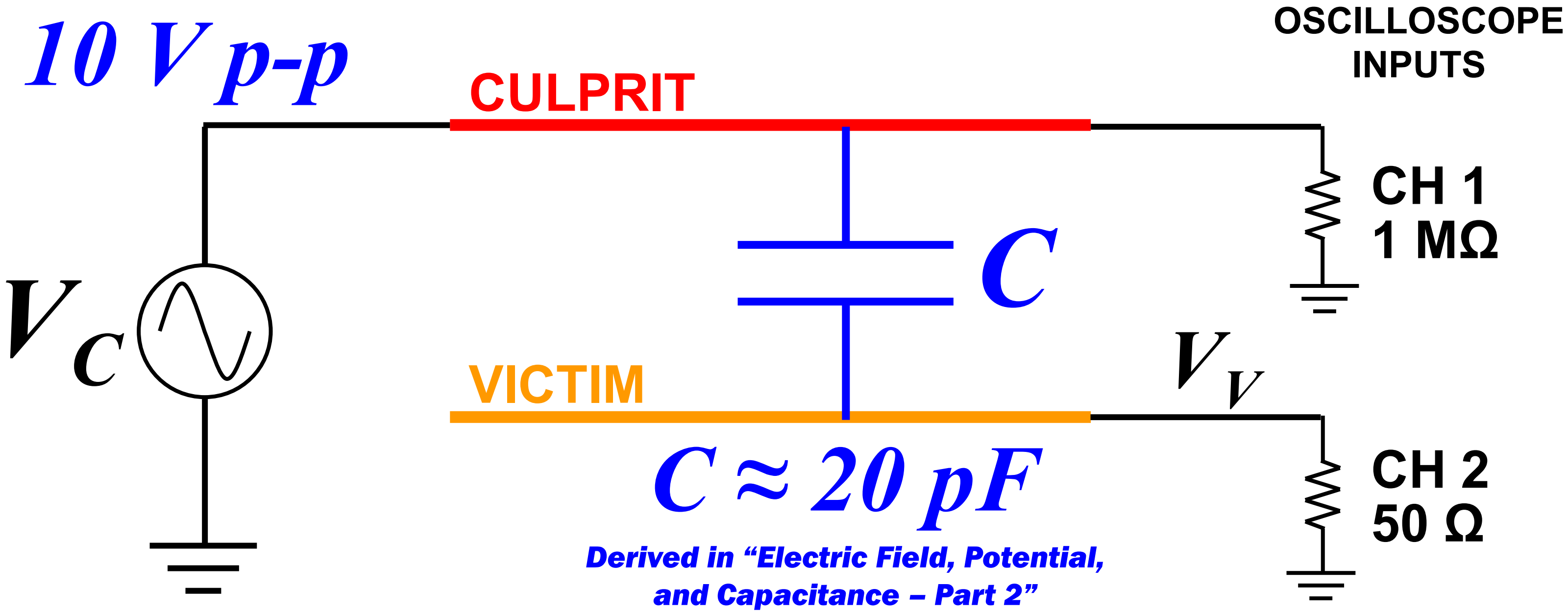
Capacitive reactance $\frac{V}{I_D} = X_C = \frac{1}{j\omega C} = \frac{1}{j2\pi f C}$ ***…decreases with frequency***

Virtual Demonstration: Capacitive Coupling

Details in "Fundamentals of Electromagnetics" video: Electric Field, Potential, and Capacitance – Part 1



Virtual Demonstration: Capacitive Coupling (cont.)



$C \approx 20\ pF$

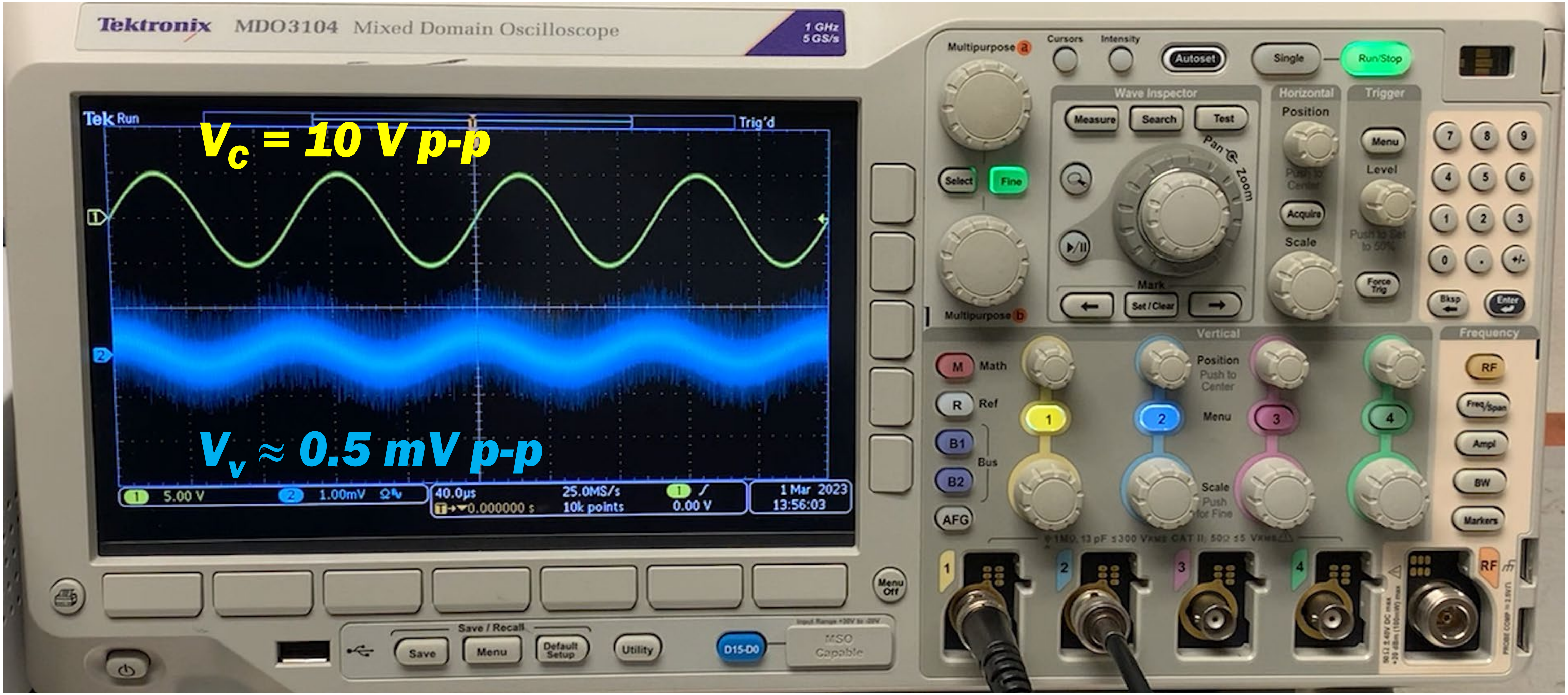
Derived in "Electric Field, Potential, and Capacitance – Part 2"

$$X_C = \frac{1}{j\omega C} = \frac{1}{j2\pi f C} \qquad V_V = V_C \cdot \frac{50\ \Omega}{\sqrt{(50\ \Omega)^2 + (X_C)^2}}$$

$$X_C(10\ kHz) \approx 800\ k\Omega \qquad V_V(10\ kHz) \approx 0.6\ mV$$

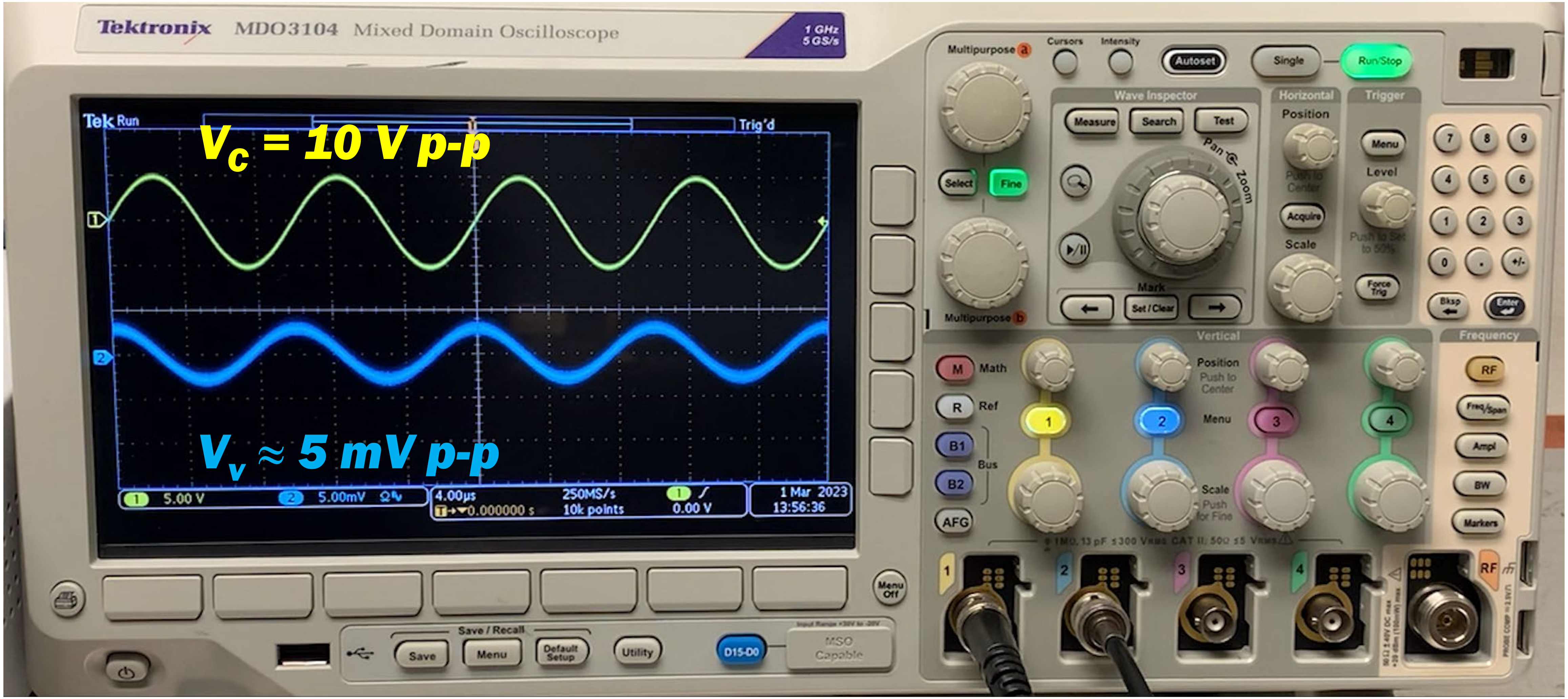
Virtual Demonstration: Capacitive Coupling (cont.)

f = 10 kHz



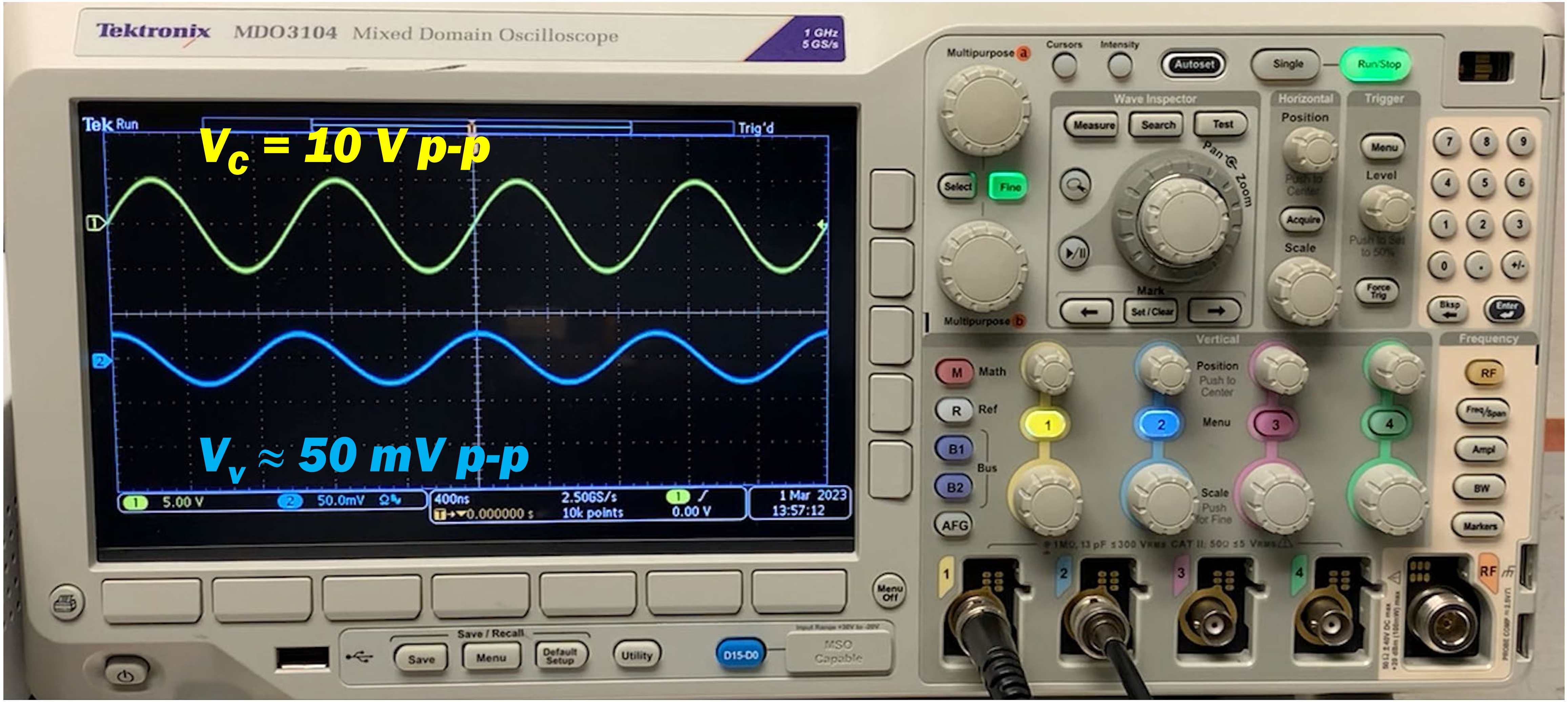
Virtual Demonstration: Capacitive Coupling (cont.)

***f* = 100 kHz**

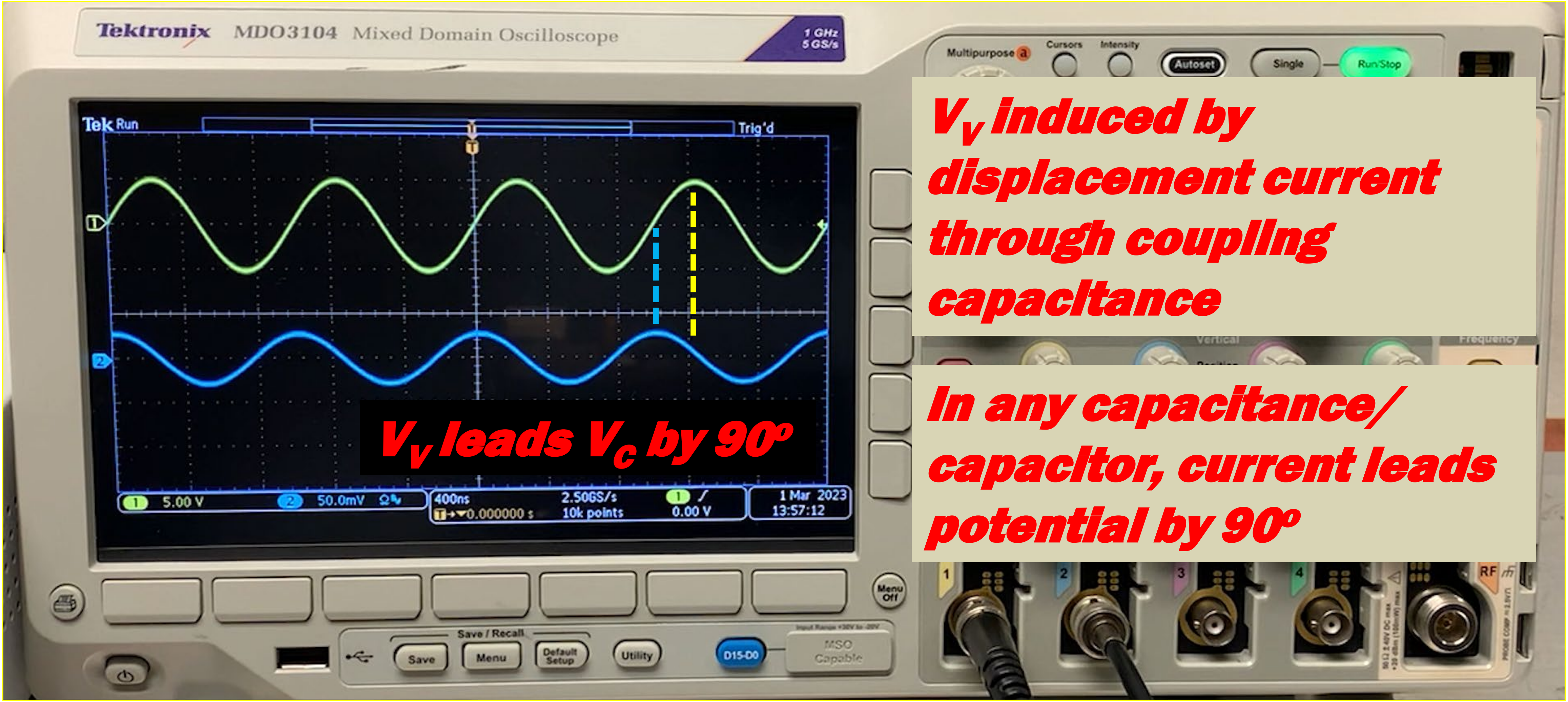


Virtual Demonstration: Capacitive Coupling (cont.)

f = 1 MHz



Virtual Demonstration: Capacitive Coupling (cont.)



V_V leads V_C by 90°

V_V induced by displacement current through coupling capacitance

In any capacitance/ capacitor, current leads potential by 90°

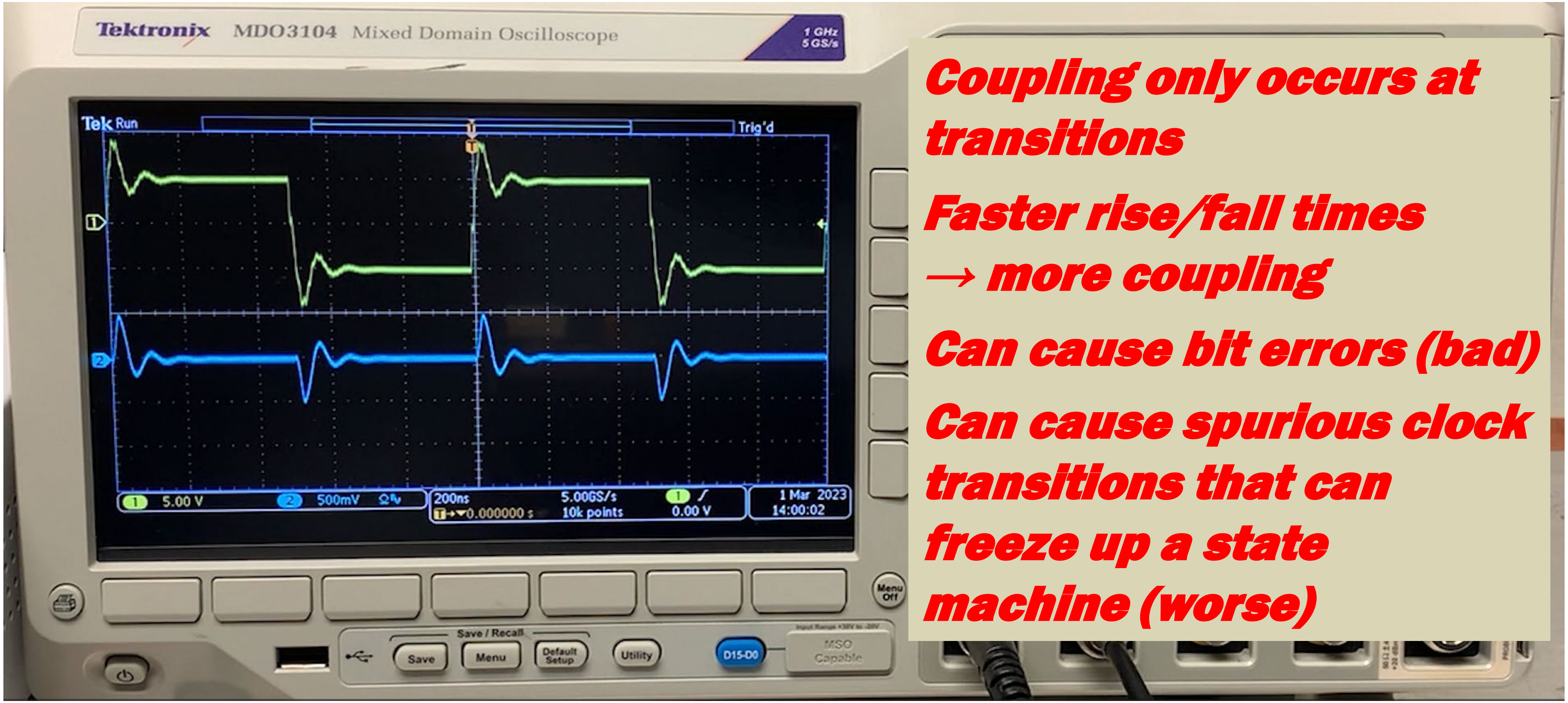
Is this amount of coupling a problem?

It depends…

"But I'm a digital logic designer,
and I only care about 1's and 0's.
What do I care about any of this?"

Uh, well…

Virtual Demonstration: Capacitive Coupling (cont.)



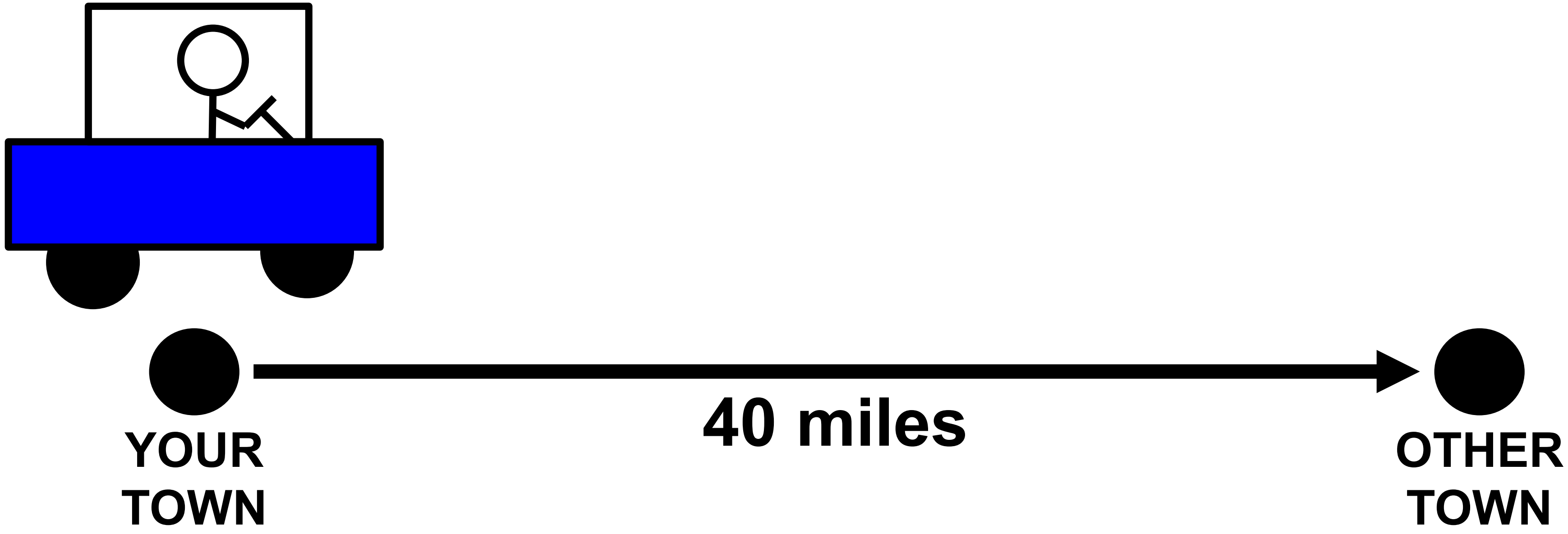
Coupling only occurs at transitions

Faster rise/fall times → more coupling

Can cause bit errors (bad)

Can cause spurious clock transitions that can freeze up a state machine (worse)

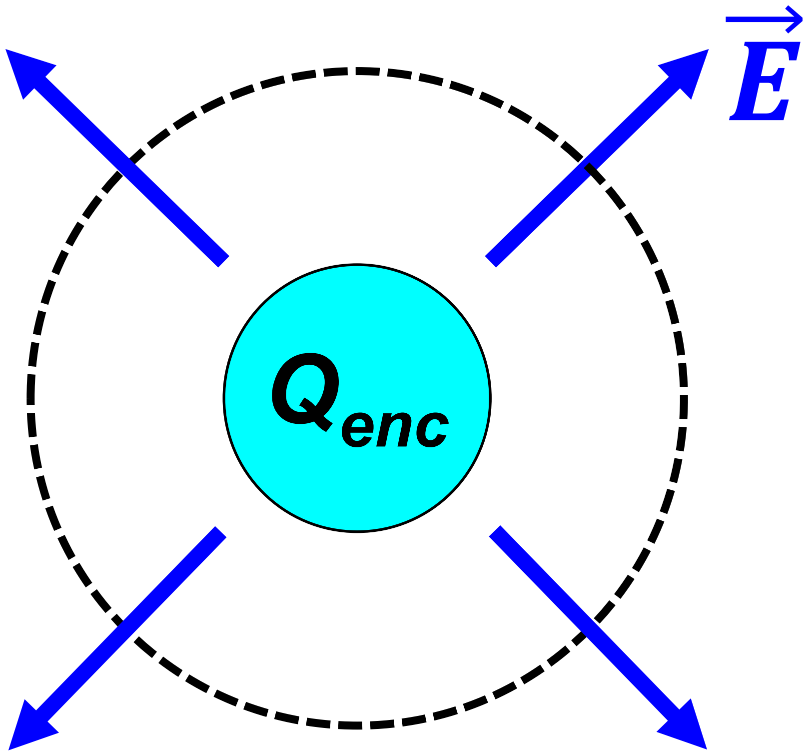
Working Models and Math Example



$$\underbrace{40\text{ miles} \times \frac{1\text{ hour}}{60\text{ miles}} \times \frac{60\text{ minutes}}{1\text{ hour}}}_{\textit{40 minutes (assuming expressway))}} + (2 \times 10\text{ minutes})$$

$$= 60\text{ minutes}$$

$$\approx \mathbf{1\ hour}$$



"Electric flux" = Q_{enc}

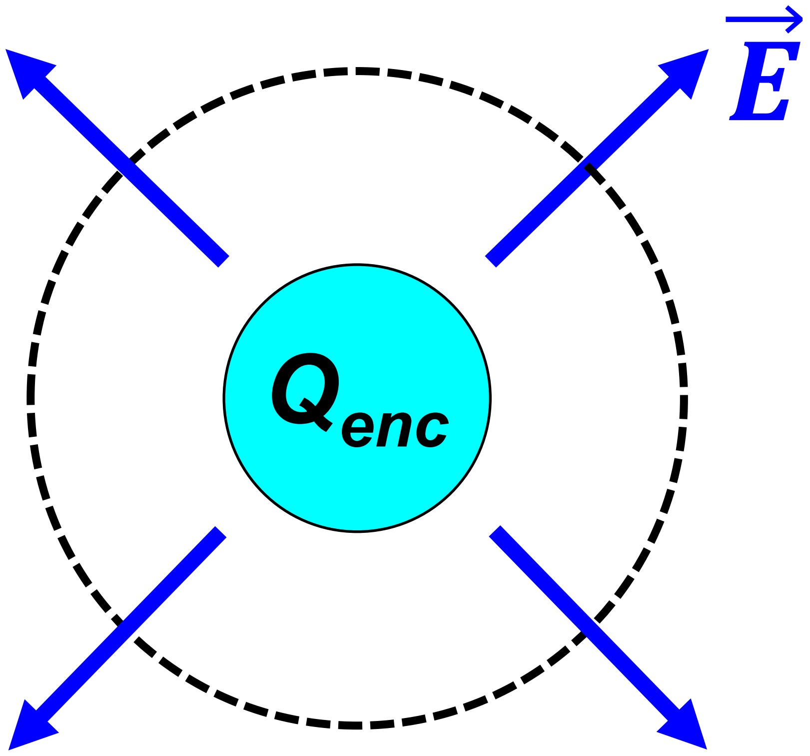
Enclosed charge produces electric field

Electric flux density:

$$\vec{D} = \frac{flux}{area}$$

$$= \varepsilon \vec{E}$$

Electric Flux Density



Electric flux density units:

$$\varepsilon \vec{E}$$

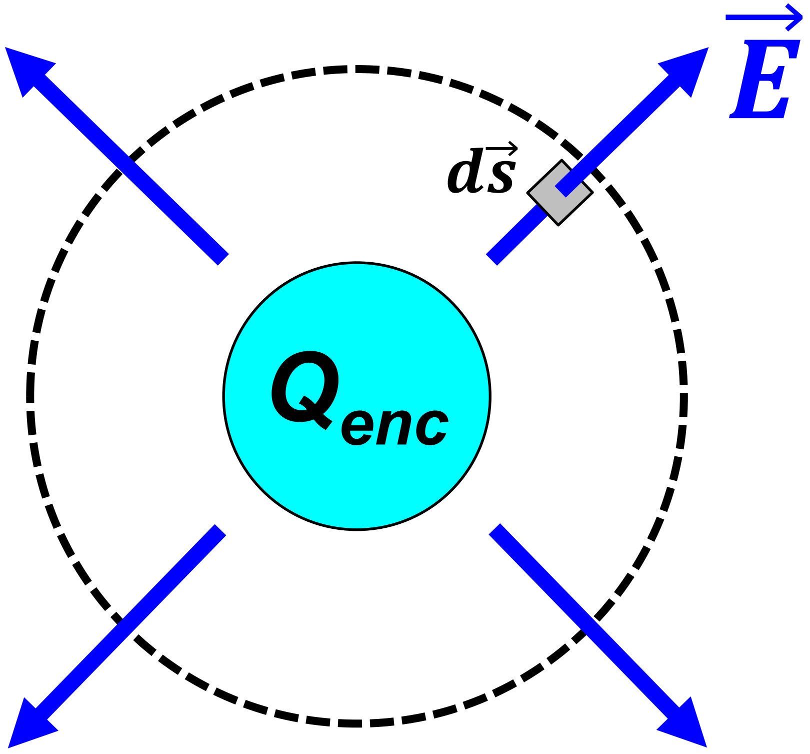
$$\frac{F}{m} \cdot \frac{V}{m}$$

$$= \frac{F \cdot V}{m^2} \qquad C = \frac{Q}{V} \qquad Q = CV$$

$$= \frac{Coulombs}{m^2}$$

Charge per unit area

Gauss's Law for Electric Field



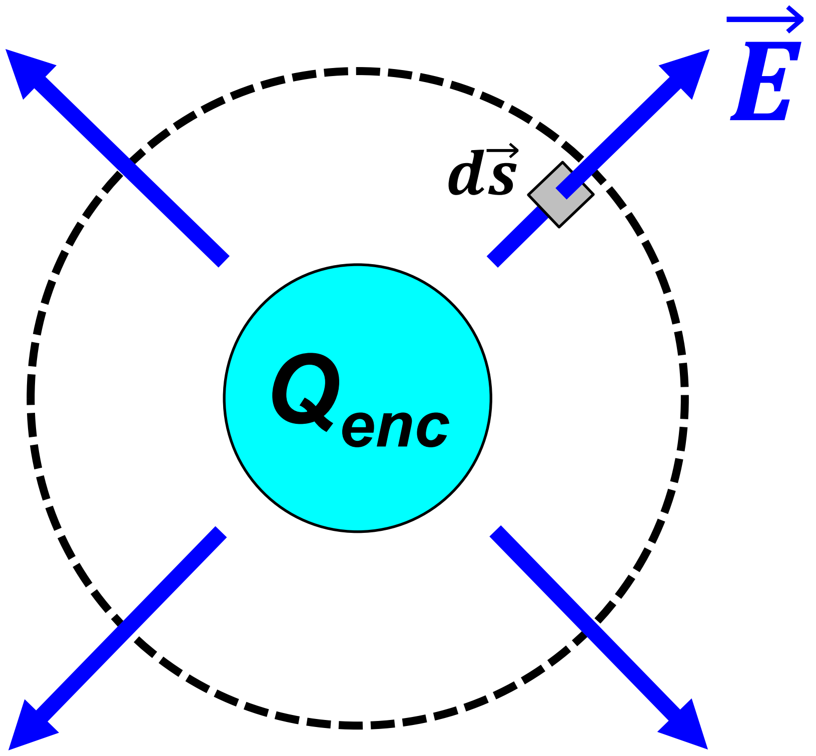
Total electric flux:

$$\oiint \varepsilon \vec{E} \cdot d\vec{s} = Q_{enc}$$

$$\oiint \vec{E} \cdot d\vec{s} = \frac{Q_{enc}}{\varepsilon}$$

Gauss's Law for electric field

Gauss's Law: Integral and Differential Forms



Integral form:

$$\oiint \vec{E} \cdot d\vec{s} = \frac{Q_{enc}}{\varepsilon}$$

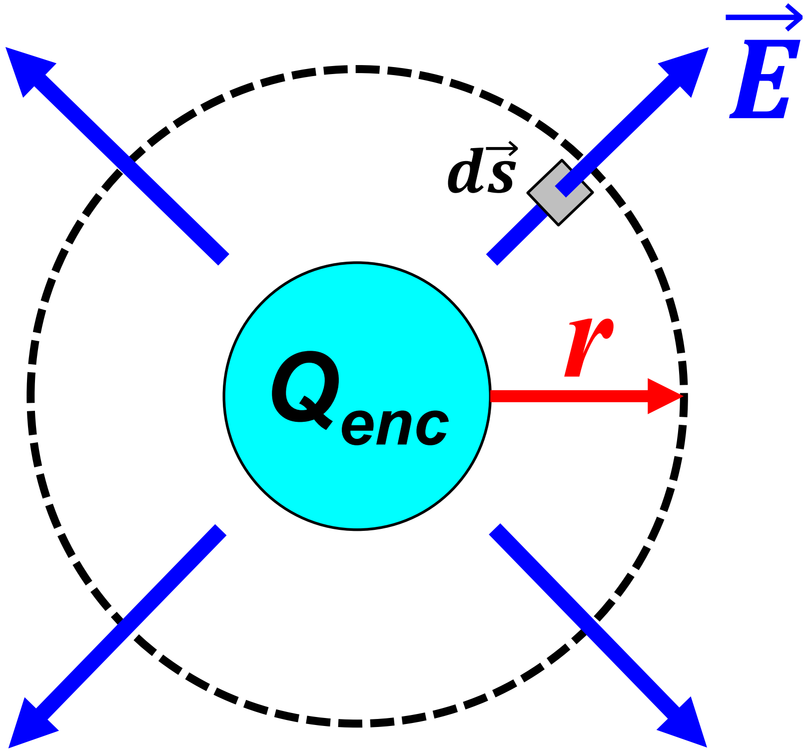
Differential form:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon}$$

$\rho = \text{volume charge density}$

NOTE: Personally, I generally prefer the integral form,
but you'll need to be able to recognize both forms

Gauss's Law: Spherical Charge Example



$$\oiint \vec{E} \cdot d\vec{s} = \frac{Q_{enc}}{\varepsilon}$$

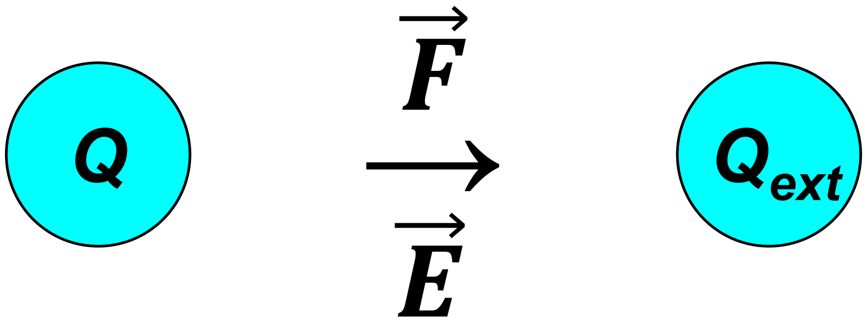
For spherical charge, surface can be defined as sphere of radius r:

$$E \cdot 4\pi r^2 = \frac{Q_{enc}}{\varepsilon}$$

$$E = \frac{Q_{enc}}{4\pi\varepsilon r^2}$$

Electric field from small spherical charge

Electric Field and Force Revisited: Coulomb's Law



$$\vec{E} = \frac{\vec{F}}{Q_{ext}} \qquad \vec{F} = \vec{E} Q_{ext}$$

$$E = \frac{Q_{enc}}{4\pi\varepsilon r^2} \qquad F = \frac{Q Q_{ext}}{4\pi\varepsilon r^2}$$

Coulomb's Law*

***Discovered experimentally by Coulomb ~50 years before Gauss published his more general law in 1835**

Electric Field, Potential, and Capacitance

$$\oiint \vec{\boldsymbol{E}} \cdot \boldsymbol{d}\vec{\boldsymbol{s}} = \frac{Q_{enc}}{\varepsilon}$$

Use Gauss's Law to calculate electric field as function of distance from charge

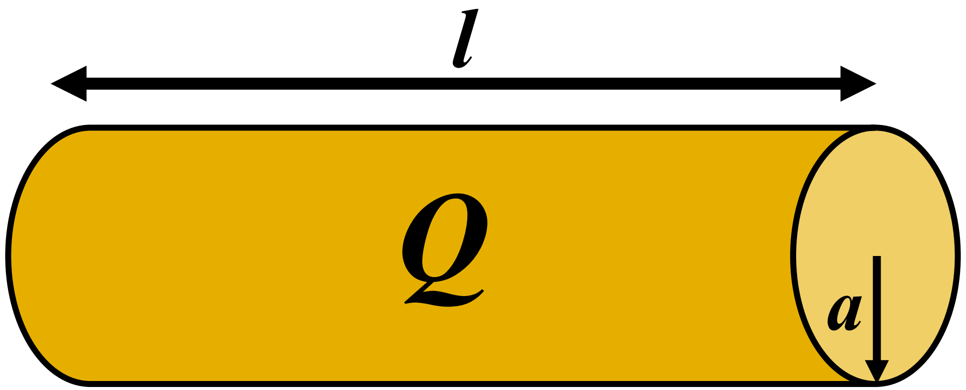
$$V = -\int_{P1}^{P2} \vec{\boldsymbol{E}} \cdot \boldsymbol{d}\vec{\boldsymbol{l}}$$

Take negative of line integral of electric field to obtain potential difference between two points (function of distance and charge)

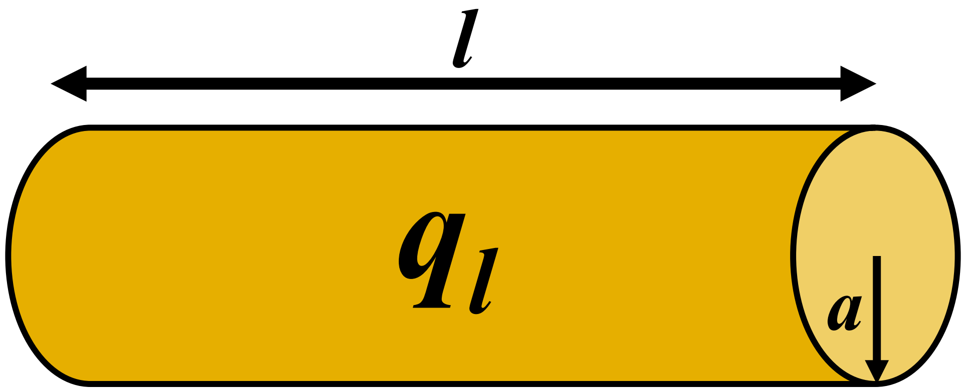
$$C = \frac{Q}{V}$$

Divide charge by potential to obtain capacitance

Charged Cylinder



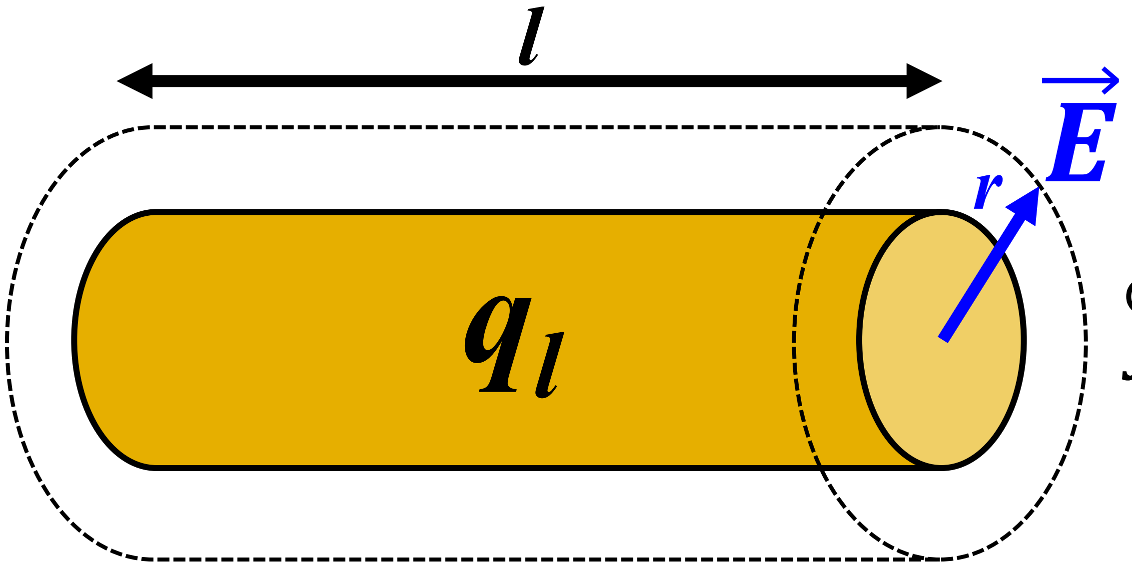
Charged Cylinder (cont.)



Linear charge density

$$\frac{Q}{l} = q_l$$

Charged Cylinder (cont.)



$$\oint \vec{E} \cdot d\vec{s} = \frac{Q_{enc}}{\varepsilon}$$

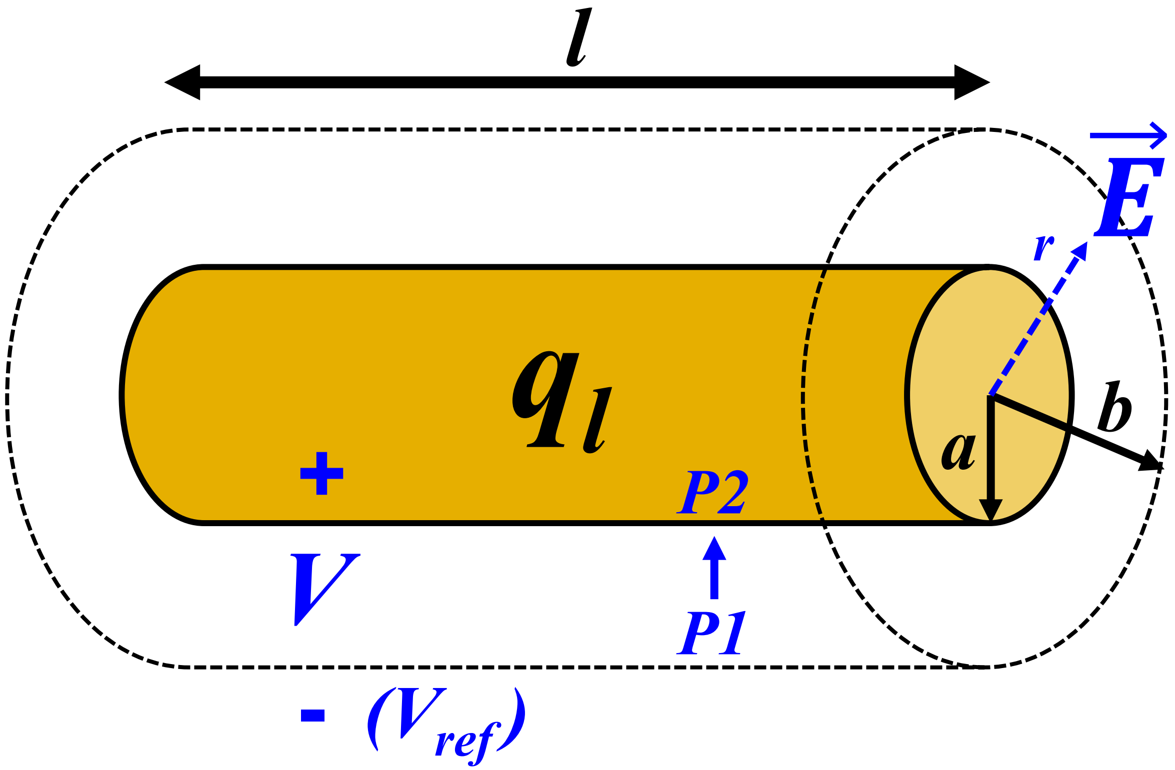
$$E \cdot \underbrace{2\pi r l}_{\text{Surface area of cylinder}} = \frac{q_l l}{\varepsilon}$$

$$E \cdot 2\pi r = \frac{q_l}{\varepsilon}$$

$$E = \frac{q_l}{2\pi\varepsilon r}$$

Charged Cylinder (cont.)

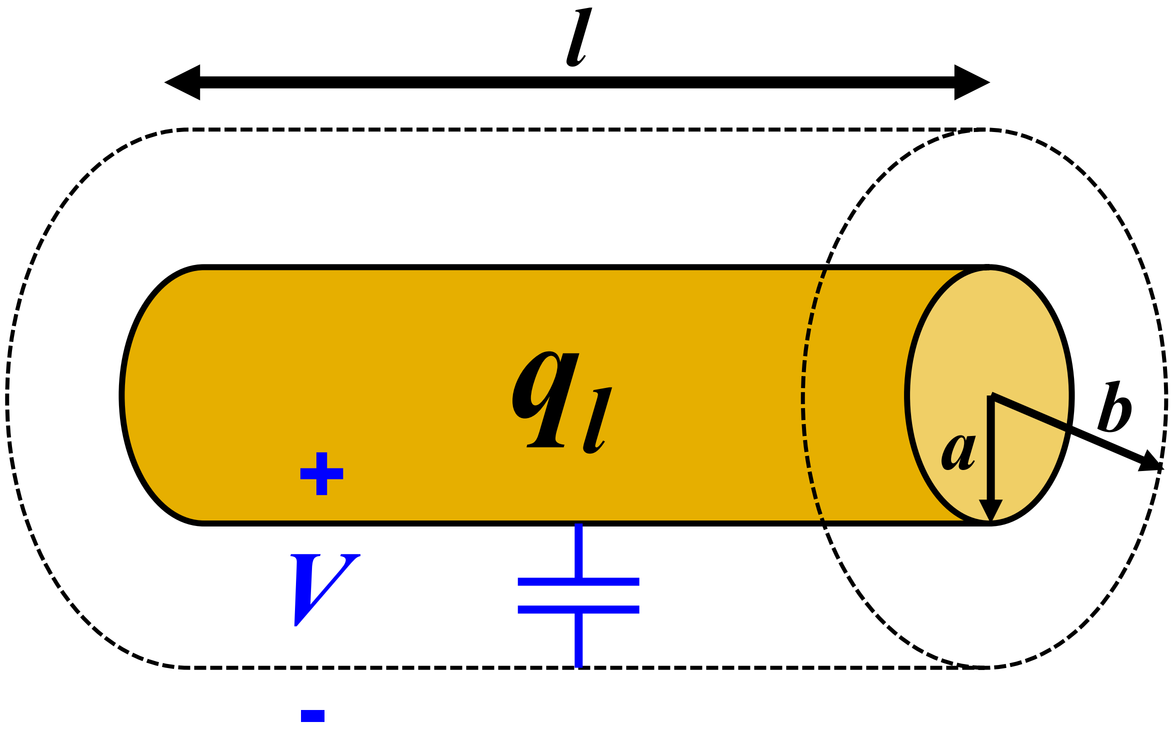
$$V = -\int_{P1}^{P2} \vec{E} \cdot d\vec{l}$$



$$E = \frac{q_l}{2\pi\varepsilon r}$$

$$V = -\int_{b}^{a} \frac{q_l}{2\pi\varepsilon r} dr \rightarrow V = \int_{a}^{b} \frac{q_l}{2\pi\varepsilon r} dr \rightarrow V = \frac{q_l}{2\pi\varepsilon} \int_{a}^{b} \frac{1}{r} dr \rightarrow V = \frac{q_l}{2\pi\varepsilon} \ln\left(\frac{b}{a}\right)$$

Charged Cylinder (cont.)



$$V = \frac{q_l}{2\pi\varepsilon}\ln\left(\frac{b}{a}\right)$$

$$C = \frac{Q}{V}$$

Capacitance per unit length:

$$C_l = \frac{q_l}{V}$$

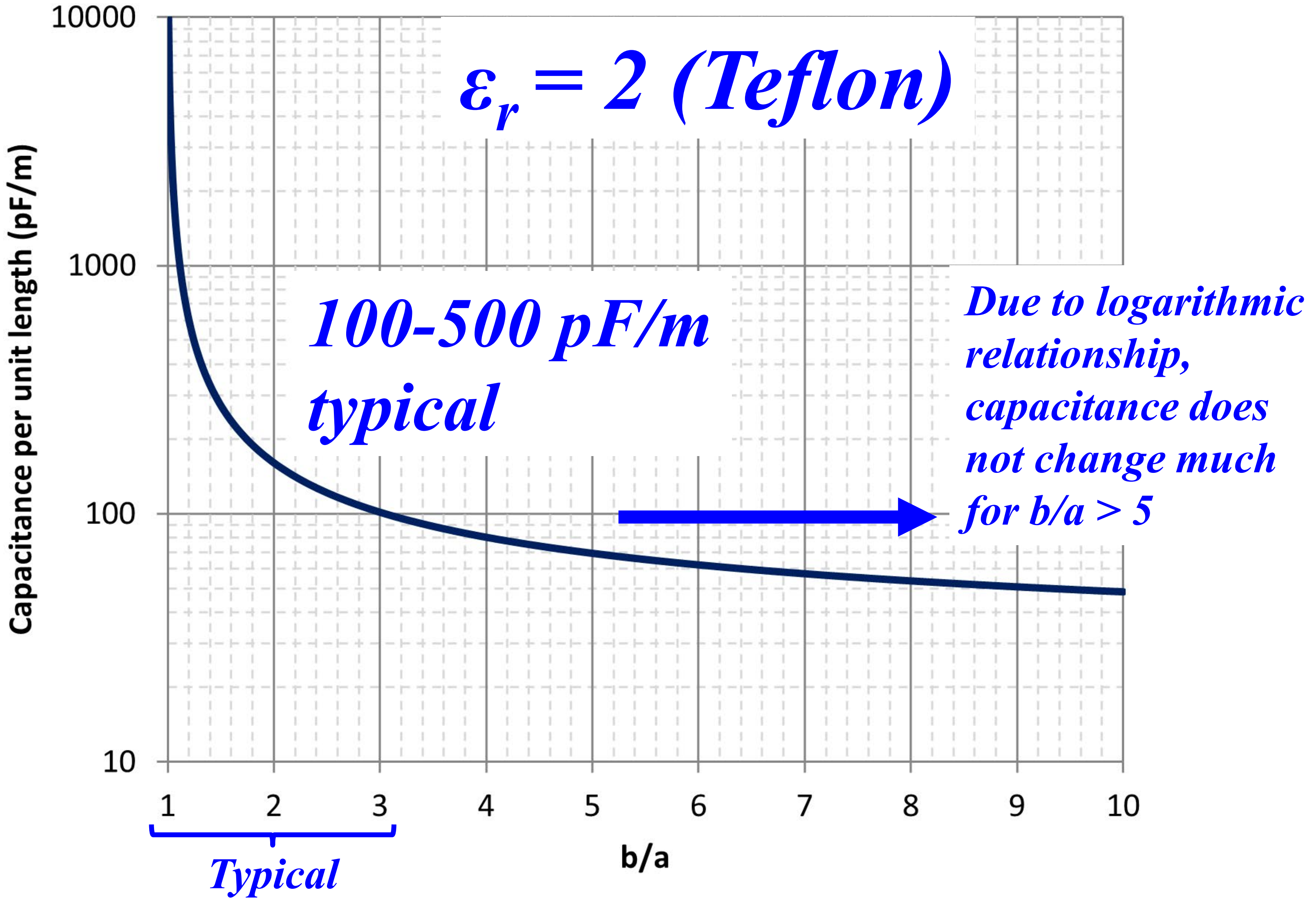
$$C_l = \frac{2\pi\varepsilon}{\ln\left(\frac{b}{a}\right)}$$

Capacitance per unit length of coaxial cable

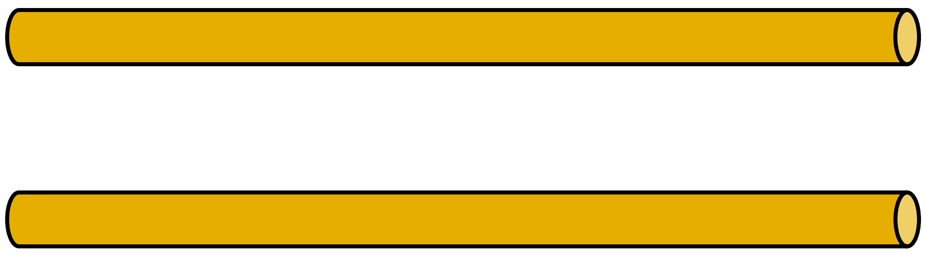
Coaxial Cable

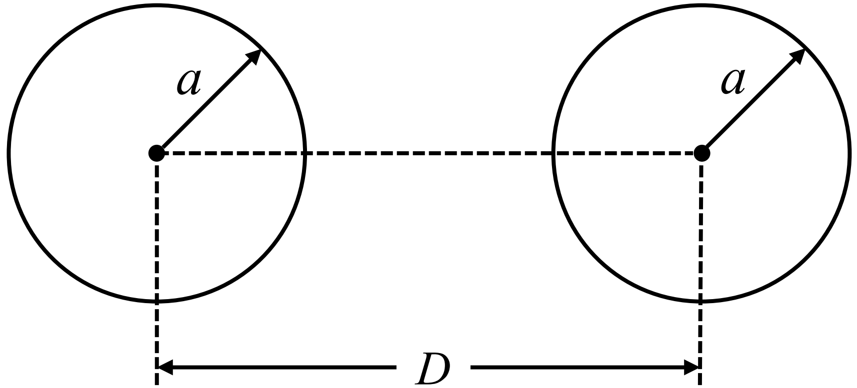
$$C_l = \frac{2\pi\varepsilon}{\ln\left(\frac{b}{a}\right)}$$

$$\varepsilon = \varepsilon_r \varepsilon_0$$



Parallel Wires





Capacitance per unit length of parallel wires (derivation in backup slides)

$$C_l = \frac{\pi\varepsilon}{\ln\left[\left(\frac{D}{2a}\right) + \sqrt{\left(\frac{D}{2a}\right)^2 - 1}\right]}$$

$$\ln\left[x + \sqrt{x^2 - 1}\right] = \cosh^{-1} x$$

$$C_l = \frac{\pi\varepsilon}{\cosh^{-1}\left(\frac{D}{2a}\right)}$$

(I prefer the natural log version, but you'll sometimes see the cosh version)

Parallel Wires (cont.)

$$C_l = \frac{\pi\varepsilon}{\ln\left[\left(\frac{D}{2a}\right) + \sqrt{\left(\frac{D}{2a}\right)^2 - 1}\right]}$$

$$\varepsilon = \varepsilon_r \varepsilon_0$$

Capacitive coupling demonstration:

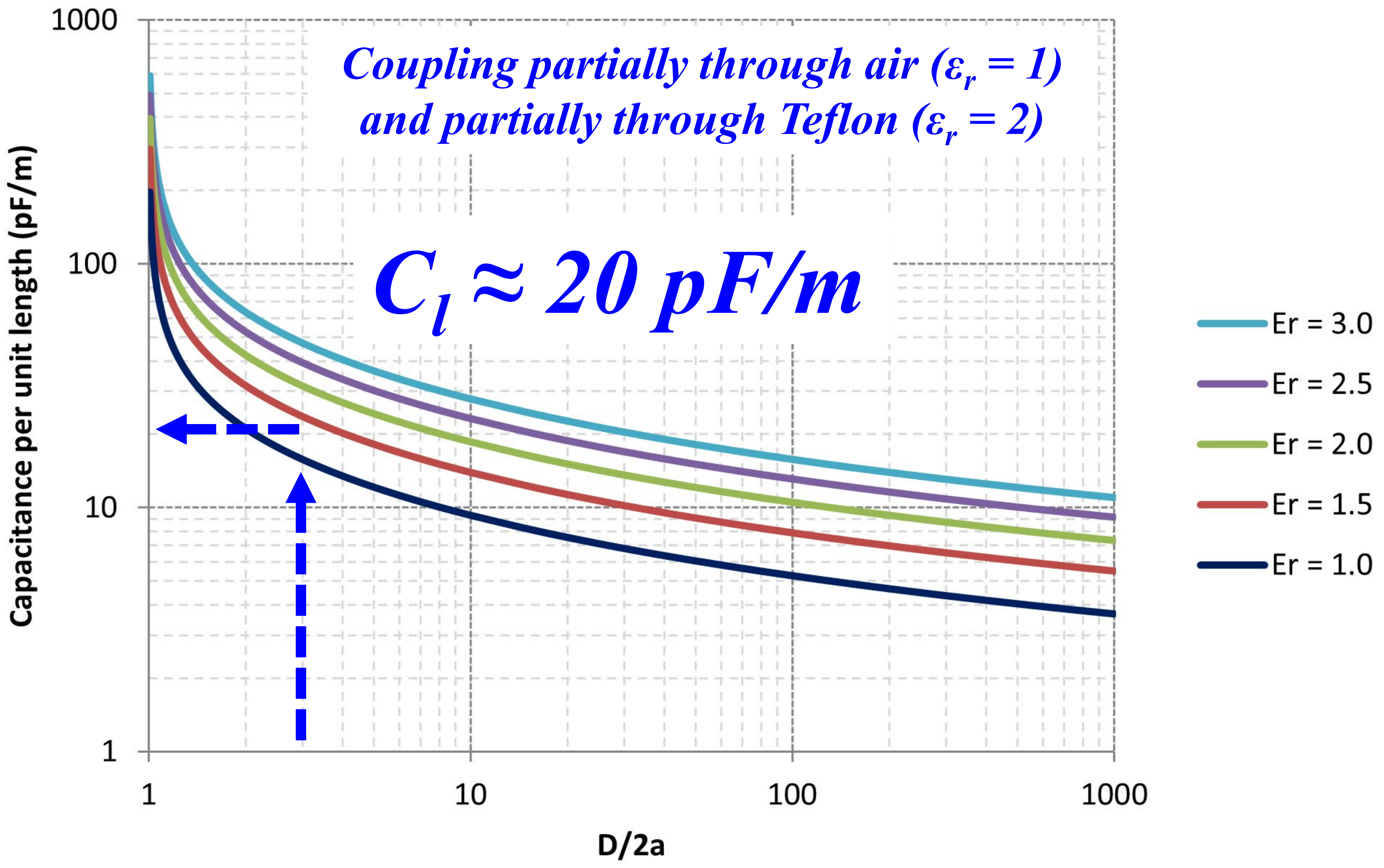
$$a \approx 0.5\ mm$$

$$D \approx 3\ mm$$

$$D/2a \approx 3$$

For 1 meter wire samples:

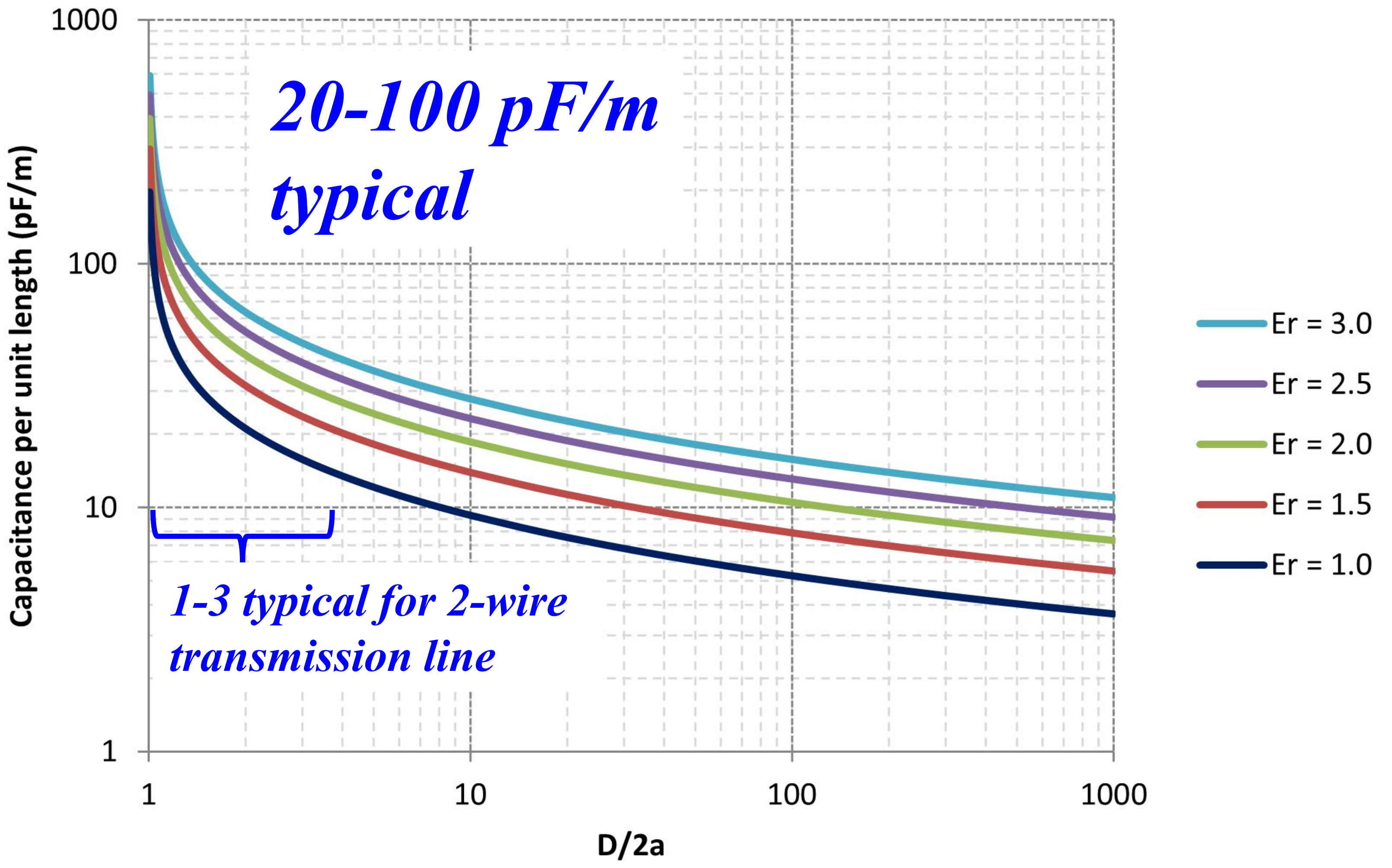
$$\boldsymbol{C \approx 20\ pF}$$



Parallel Wires / 2-Wire Transmission Line

$$C_l = \frac{\pi\varepsilon}{\ln\left[\left(\frac{D}{2a}\right) + \sqrt{\left(\frac{D}{2a}\right)^2 - 1}\right]}$$

$$\varepsilon = \varepsilon_r \varepsilon_0$$



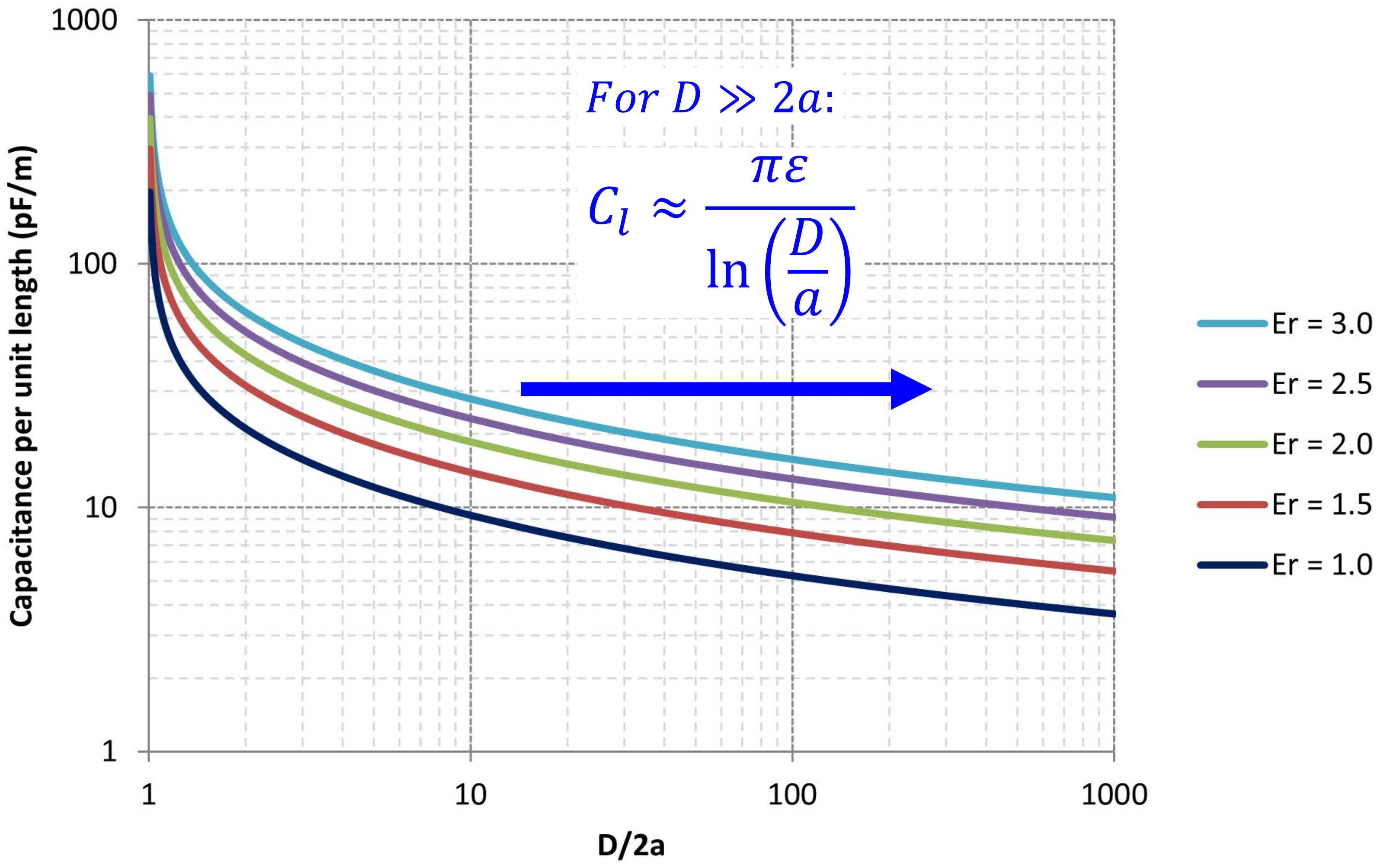
Parallel Wires / 2-Wire Transmission Line (cont.)

$$C_l = \frac{\pi\varepsilon}{\ln\left[\left(\frac{D}{2a}\right) + \sqrt{\left(\frac{D}{2a}\right)^2 - 1}\right]}$$

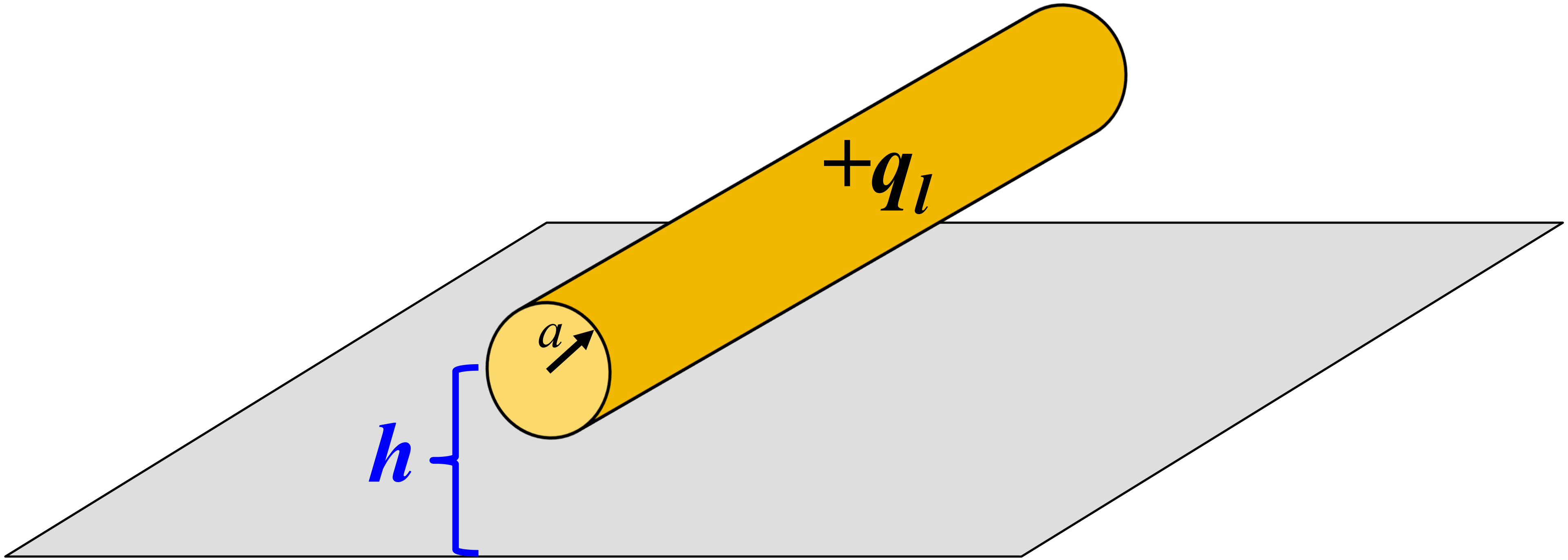
$$\varepsilon = \varepsilon_r \varepsilon_0$$

Some texts show only the simplified expression

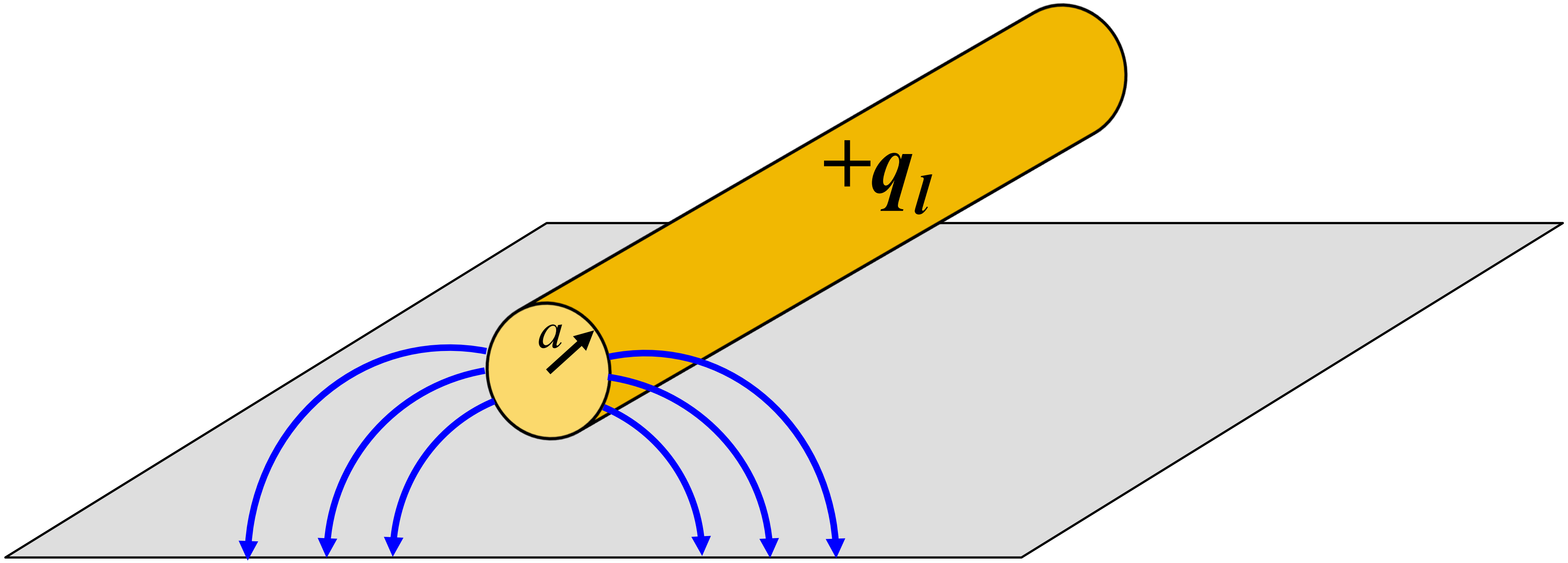
You need to know the full expression above for wires in close proximity; that is where much of our interest lies



Wire Above Ground Plane



Wire Above Ground Plane (cont.)



Wire Above Ground Plane (cont.)

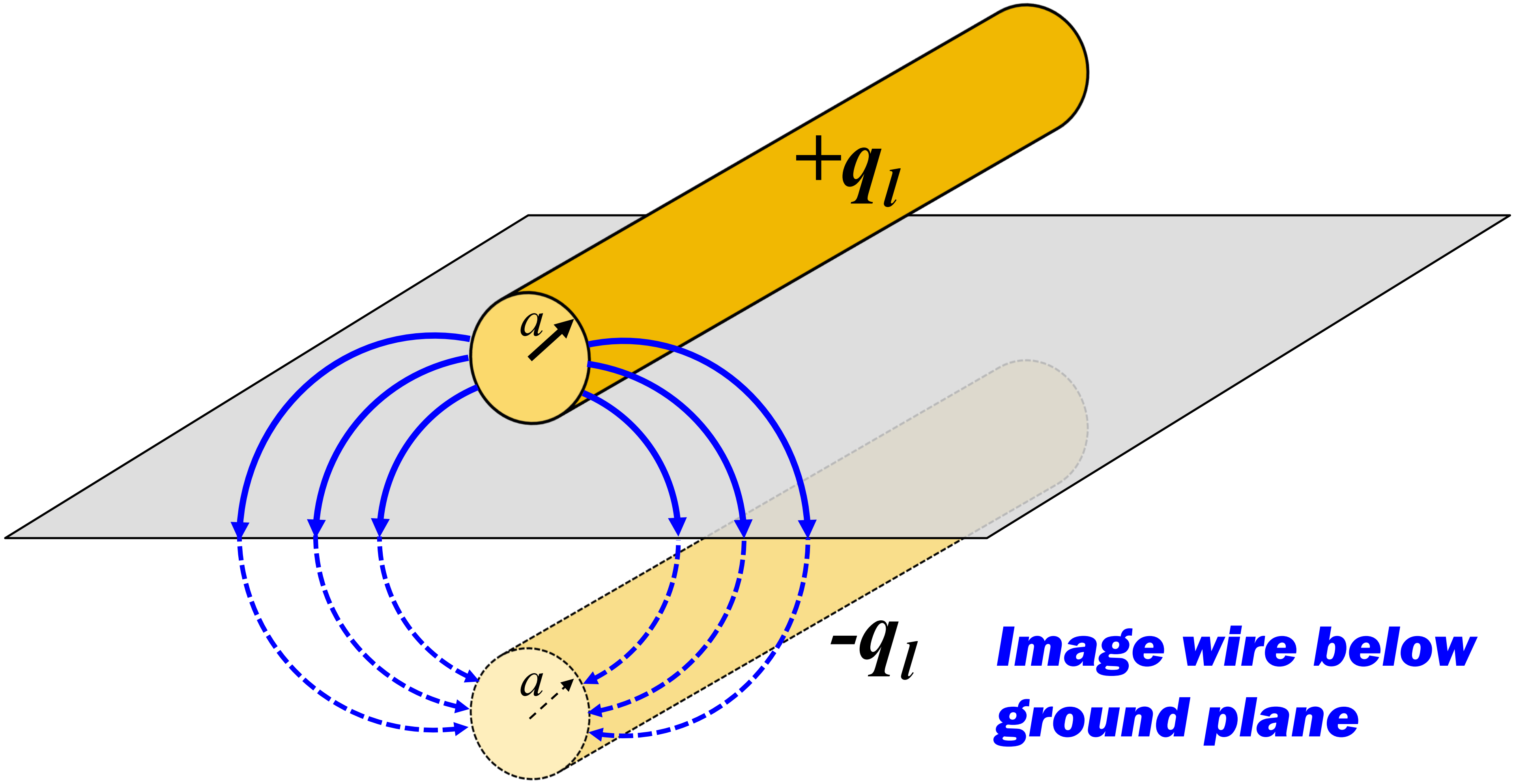
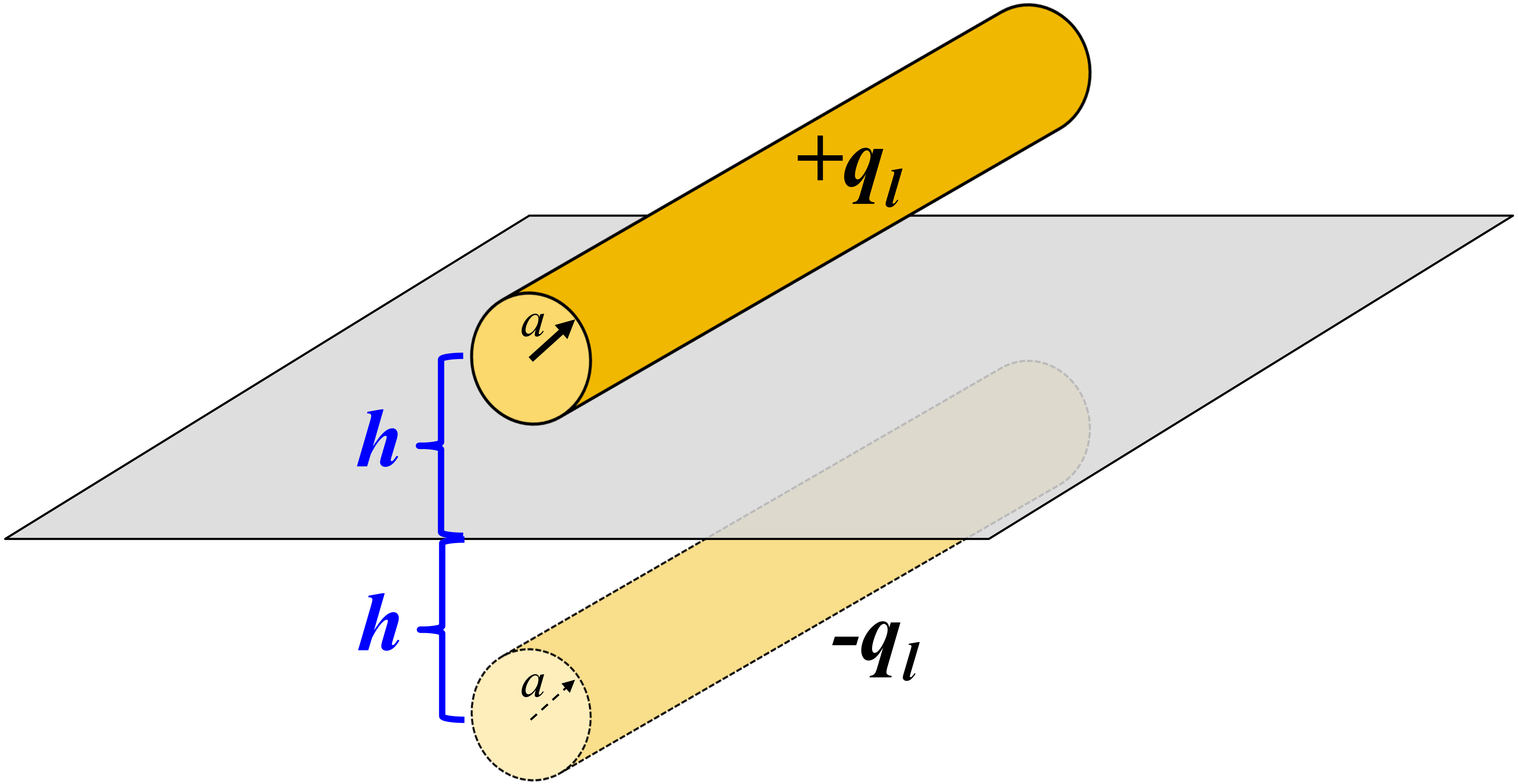
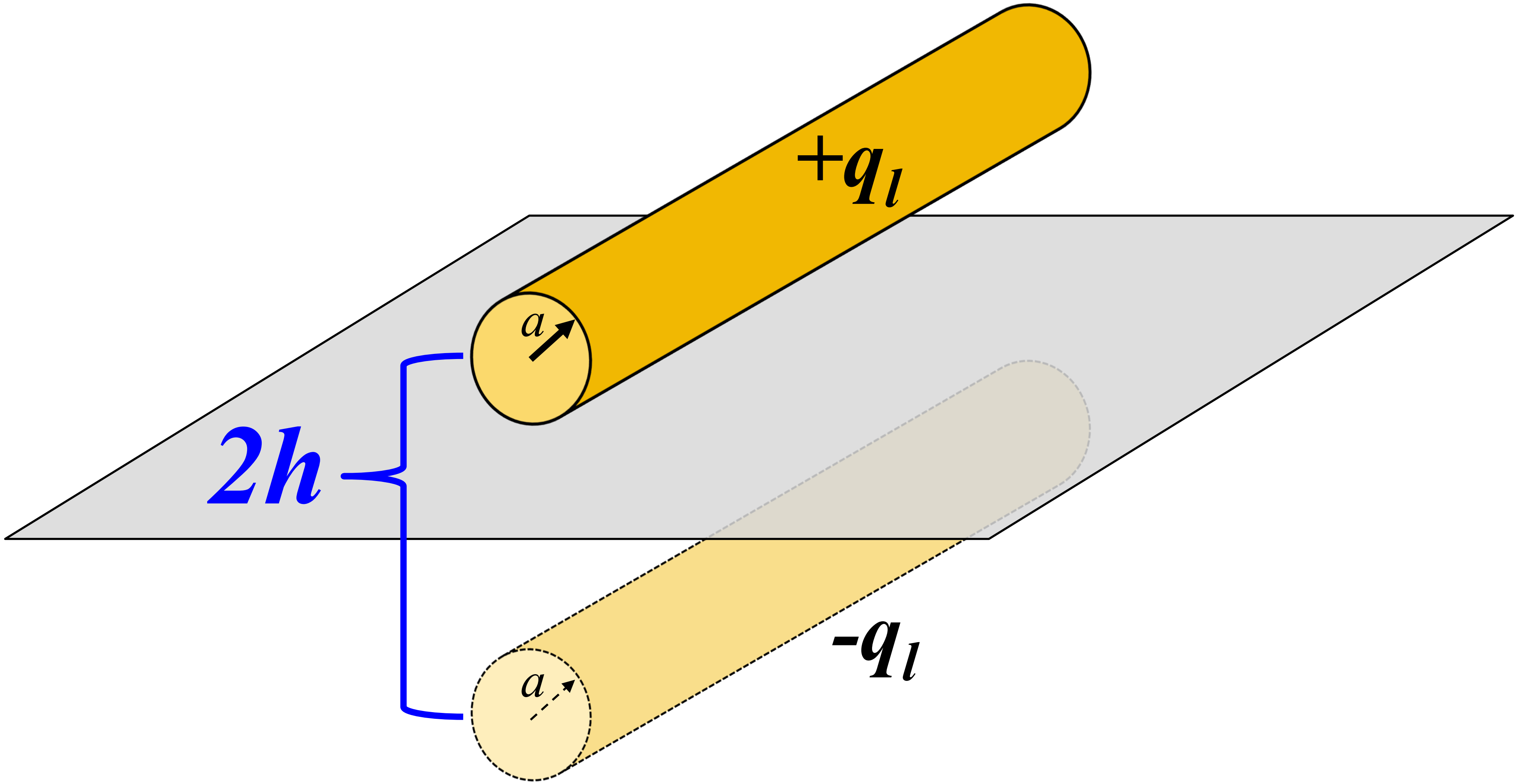


Image wire below ground plane

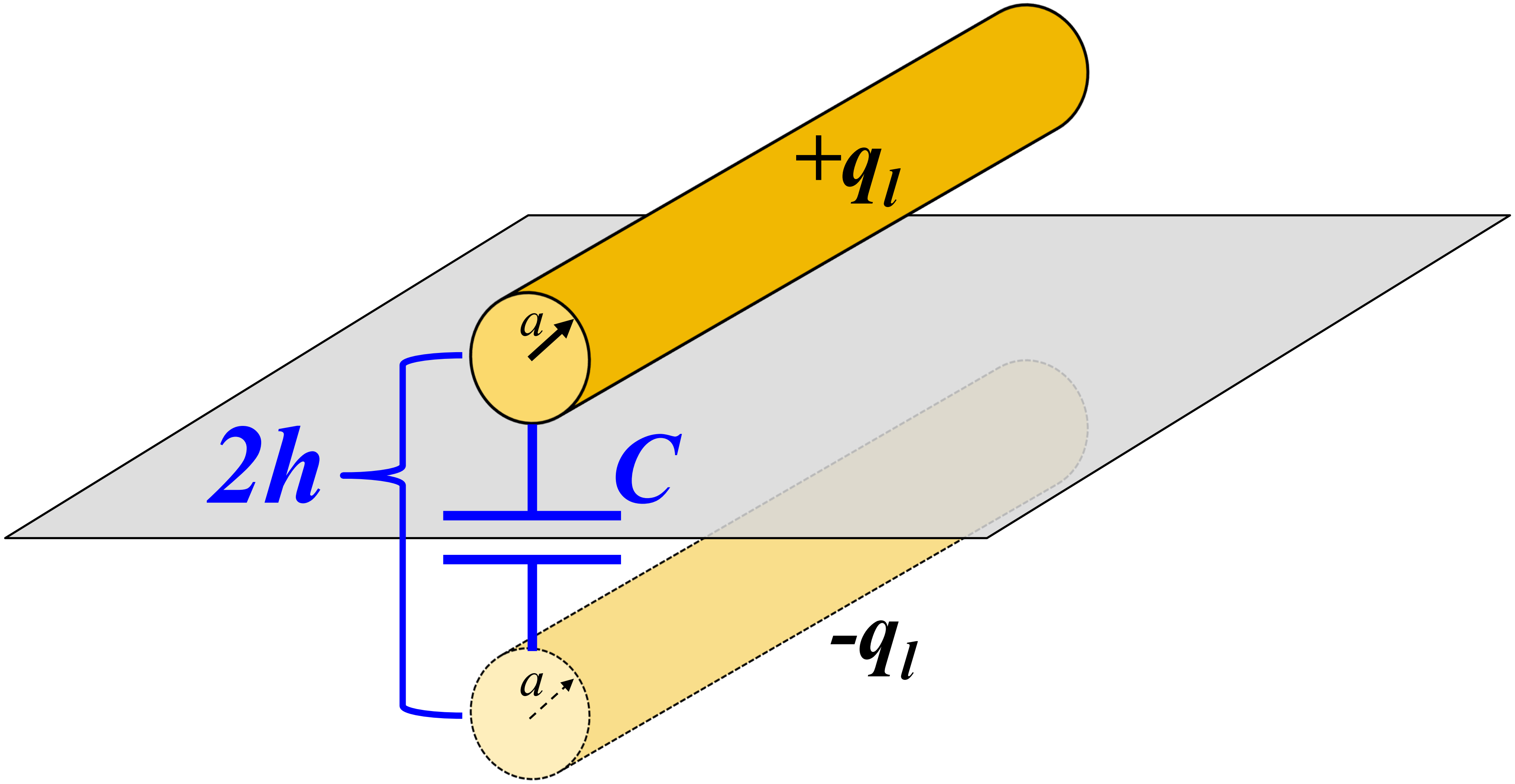
Wire Above Ground Plane (cont.)



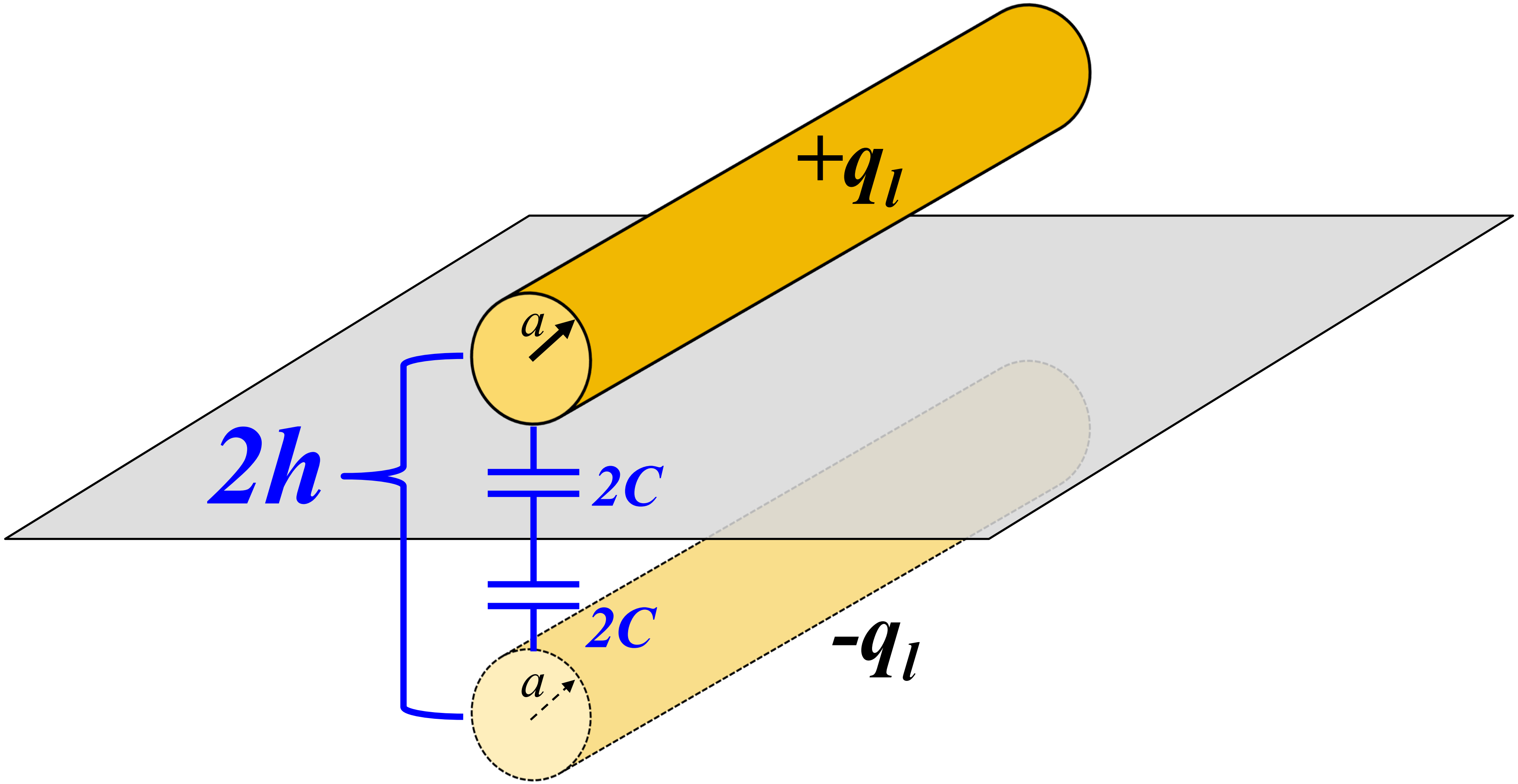
Wire Above Ground Plane (cont.)



Wire Above Ground Plane (cont.)



Wire Above Ground Plane (cont.)



Wire Above Ground Plane (cont.)

Capacitance per unit length of parallel wires:

$$C_l = \frac{\pi\varepsilon}{\ln\left[\left(\frac{D}{2a}\right) + \sqrt{\left(\frac{D}{2a}\right)^2 - 1}\right]}$$

Substitute 2h for D

Multiply entire expression by 2

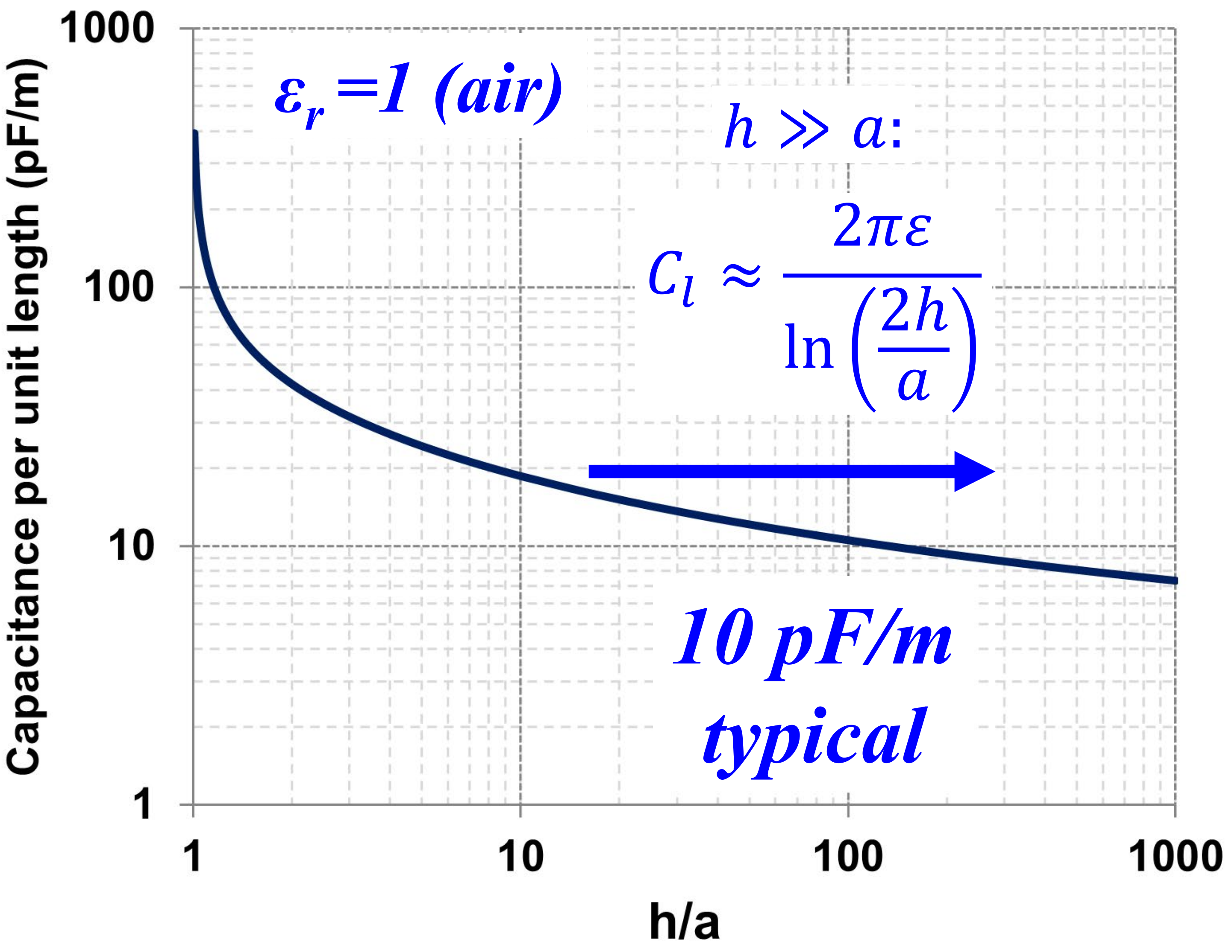
Capacitance per unit length of wire above ground plane:

$$C_l = \frac{2\pi\varepsilon}{\ln\left[\left(\frac{h}{a}\right) + \sqrt{\left(\frac{h}{a}\right)^2 - 1}\right]}$$

Wire Above Ground Plane (cont.)

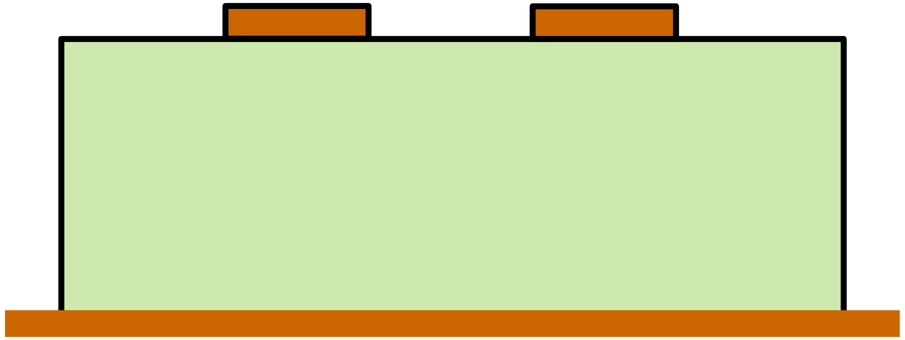
$$C_l = \frac{2\pi\varepsilon}{\ln\left[\left(\frac{h}{a}\right) + \sqrt{\left(\frac{h}{a}\right)^2 - 1}\right]}$$

$$\varepsilon = \varepsilon_r \varepsilon_0$$



Printed Circuit Boards

- General principles hold for printed circuit (PC) board traces
- Equations for wires on previous slide can give rough approximations, but...
- PC board structures are a bit more complex, and PC board modelling and simulation tools will give more accurate results



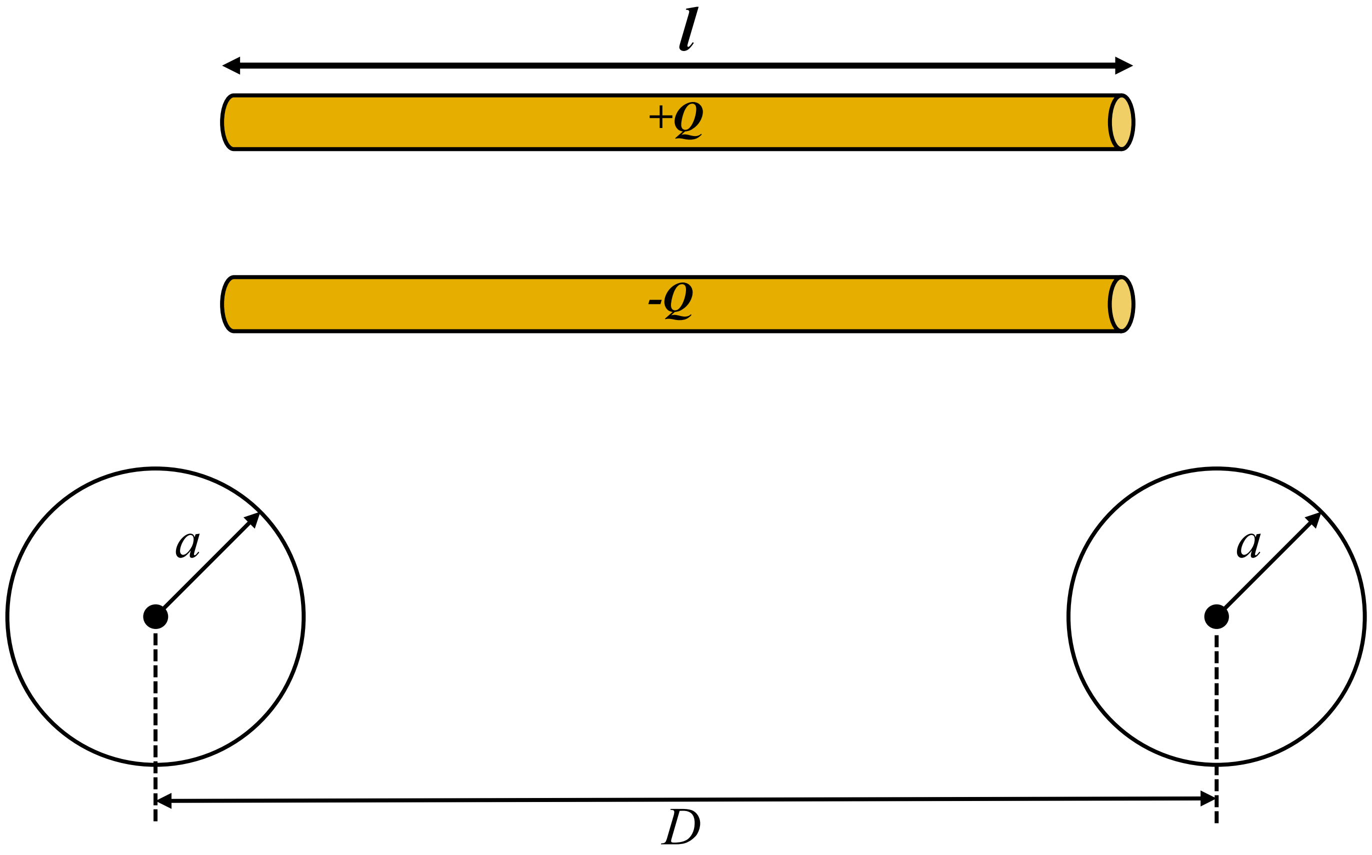
Electric Field, Potential, and Capacitance Summary

- **Electric field is directly related to familiar concepts of potential and capacitance**
- **Potential between two conductors produces capacitance whether or not it appears on your schematic**
- **Now you have a way of estimating it and understanding capacitive coupling when you see it (because you will)**
- ***Now, let's move onto magnetic fields…***

Electric Field, Potential, and Capacitance

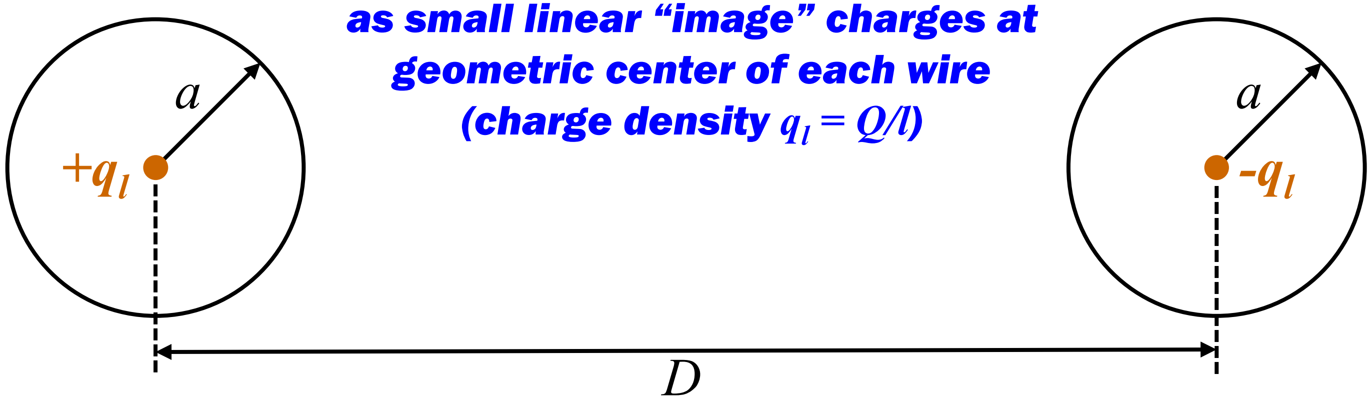
BACKUP

Parallel Wires



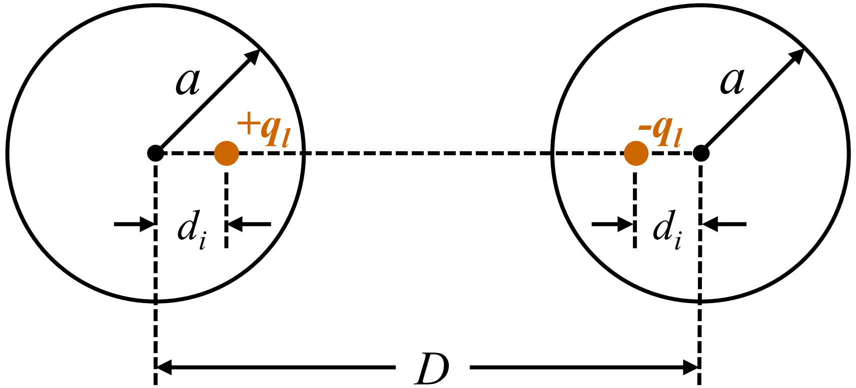
Details in "Fundamentals of Electromagnetics" video:
Electric Field, Potential, and Capacitance – Part 2

Parallel Wires (cont.)



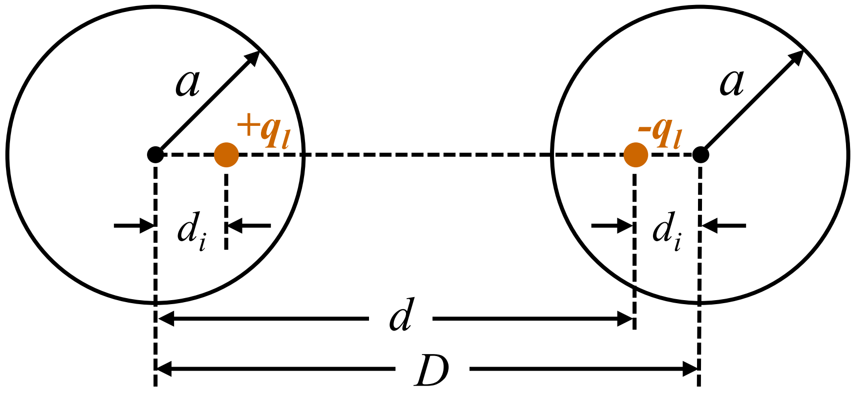
For D >> a, charges can be modelled as small linear "image" charges at geometric center of each wire (charge density $q_l = Q/l$)

Parallel Wires (cont.)



For small D, "image" charges are offset from geometric centers

Parallel Wires (cont.)

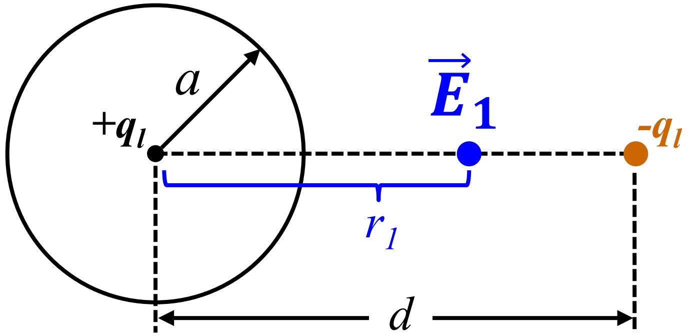


$$d = D - d_i$$

Parallel Wires (cont.)

Electric field E_1 at distance r_1 from positive charge:

$$E_1 = \frac{q_l}{2\pi\varepsilon r_1}$$

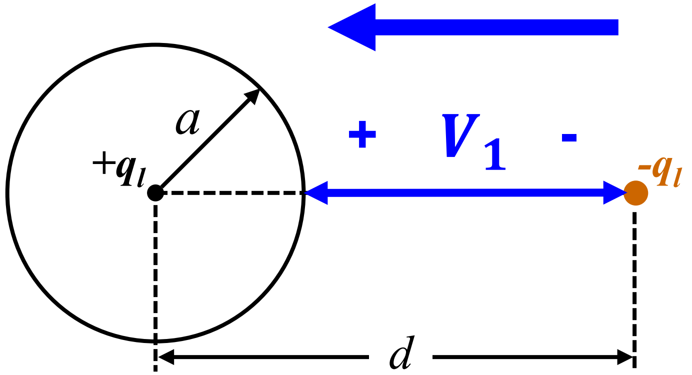


Parallel Wires (cont.)

$$E_1 = \frac{q_l}{2\pi\varepsilon r_1}$$

$$V_1 = -\int_{\boldsymbol{d}}^{\boldsymbol{a}} \vec{\boldsymbol{E}}_{\mathbf{1}} \cdot d\vec{r}_1$$

Integrate from reference potential toward potential of interest (d → a)



$$V_1 = -\int_{\boldsymbol{d}}^{\boldsymbol{a}} \frac{q_l}{2\pi\varepsilon r_1} dr_1$$

$$V_1 = \int_{\boldsymbol{a}}^{\boldsymbol{d}} \frac{q_l}{2\pi\varepsilon r_1} dr_1$$

Reverse integration limits
Cancel negative sign

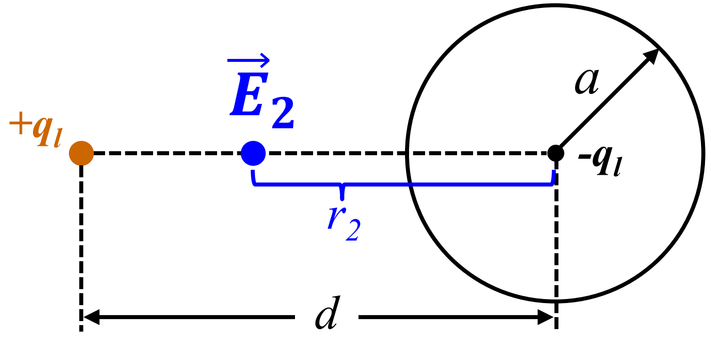
$$V_1 = \frac{q_l}{2\pi\varepsilon} \int_{\boldsymbol{a}}^{\boldsymbol{d}} \frac{1}{r_1} dr_1$$

$$V_1 = \frac{q_l}{2\pi\varepsilon} \ln\left(\frac{d}{a}\right)$$

Parallel Wires (cont.)

Electric field E_2 at distance r_2 from negative charge:

$$E_2 = \frac{-q_l}{2\pi\varepsilon r_2}$$

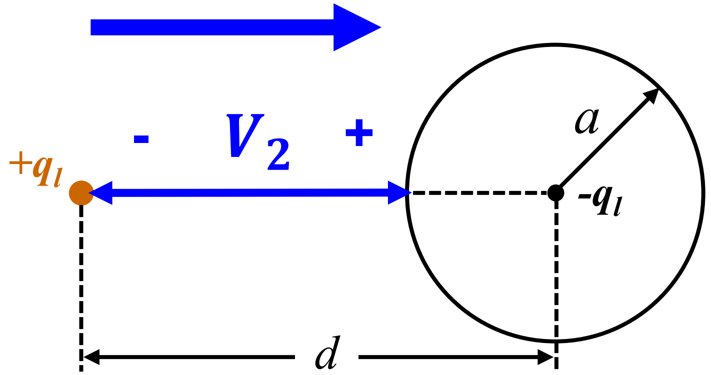


Parallel Wires (cont.)

$$E_2 = \frac{-q_l}{2\pi\varepsilon r_2}$$

$$V_2 = -\int_{d}^{a} \vec{E}_2 \cdot d\vec{r}_2$$

Integrate from reference potential toward potential of interest (d → a)



$$V_2 = -\int_{d}^{a} \frac{-q_l}{2\pi\varepsilon r_1} dr_1$$

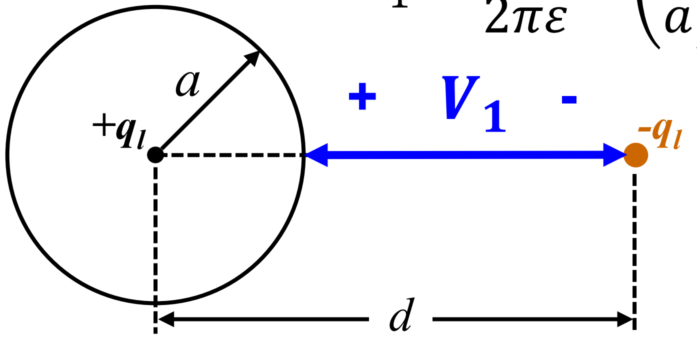
$$V_2 = \int_{a}^{d} \frac{-q_l}{2\pi\varepsilon r_1} dr_1$$

$$V_2 = \frac{-q_l}{2\pi\varepsilon} \int_{a}^{d} \frac{1}{r_1} dr_1$$

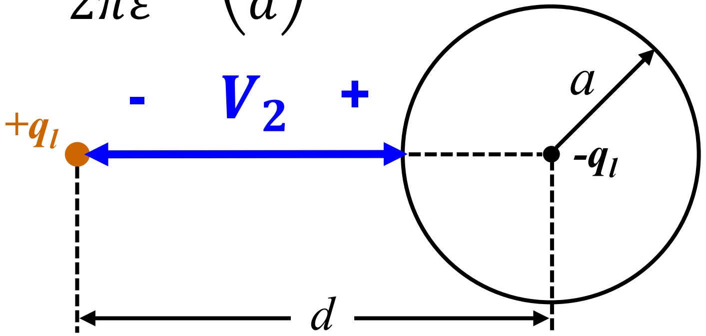
$$V_2 = \frac{-q_l}{2\pi\varepsilon} \ln\left(\frac{d}{a}\right)$$

Parallel Wires (cont.)

$$V_1 = \frac{q_l}{2\pi\varepsilon}\ln\left(\frac{d}{a}\right)$$



$$V_2 = \frac{-q_l}{2\pi\varepsilon}\ln\left(\frac{d}{a}\right)$$



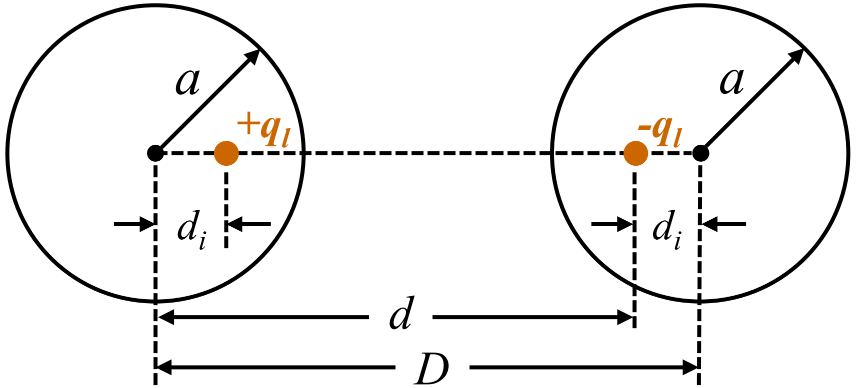
$$V_{TOT} = V_1 - V_2$$

$$V_{TOT} = \frac{q_l}{2\pi\varepsilon}\ln\left(\frac{d}{a}\right) - \left[-\frac{q_l}{2\pi\varepsilon}\ln\left(\frac{d}{a}\right)\right]$$

$$V_{TOT} = \frac{q_l}{2\pi\varepsilon}\ln\left(\frac{d}{a}\right) + \frac{q_l}{2\pi\varepsilon}\ln\left(\frac{d}{a}\right)$$

$$V_{TOT} = \frac{q_l}{\pi\varepsilon}\ln\left(\frac{d}{a}\right)$$

Parallel Wires (cont.)

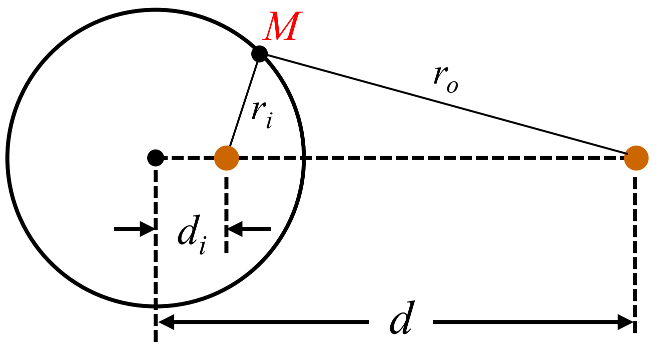


$$V_{TOT} = \frac{q_l}{\pi\varepsilon} \ln\left(\frac{d}{a}\right)$$

Need this in terms of large D

Parallel Wires (cont.)

Point M on outer surface of wire

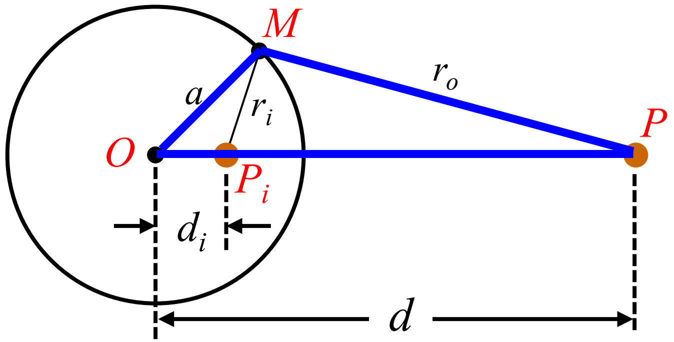


Outer surface of each wire is equipotential

Relative electric field contributions from each linear charge must be constant

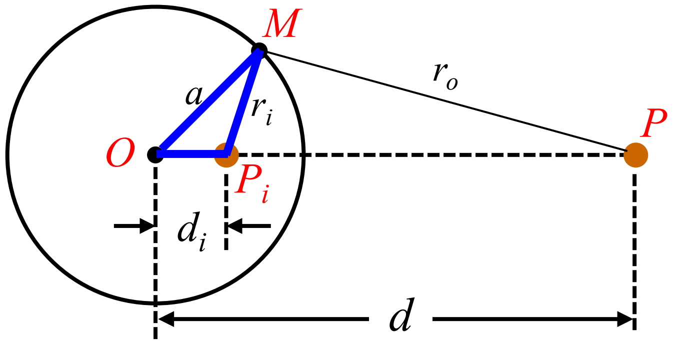
$$\frac{r_i}{r_o} = Constant$$

Parallel Wires (cont.)



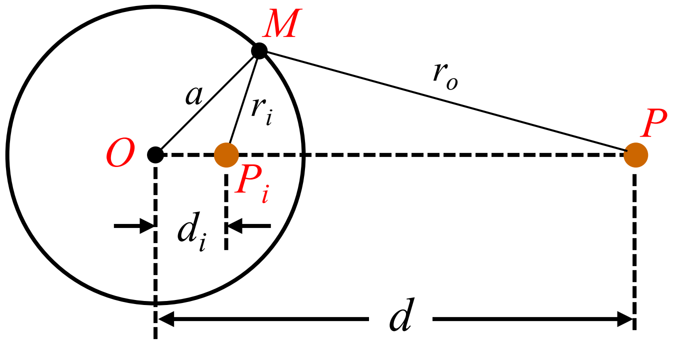
Triangle OPM...

Parallel Wires (cont.)



...must be similar to triangle OMP$_i$

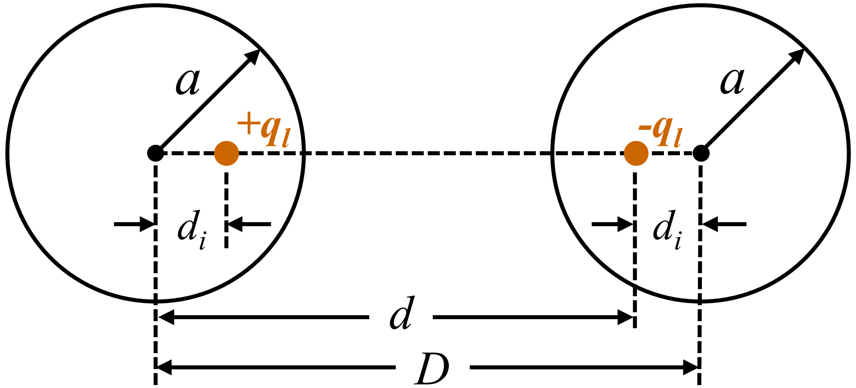
Parallel Wires (cont.)



$$\frac{r_i}{r_o} = \frac{d_i}{a} = \frac{a}{d} = Constant$$

$$d_i = \frac{a^2}{d}$$

Parallel Wires (cont.)

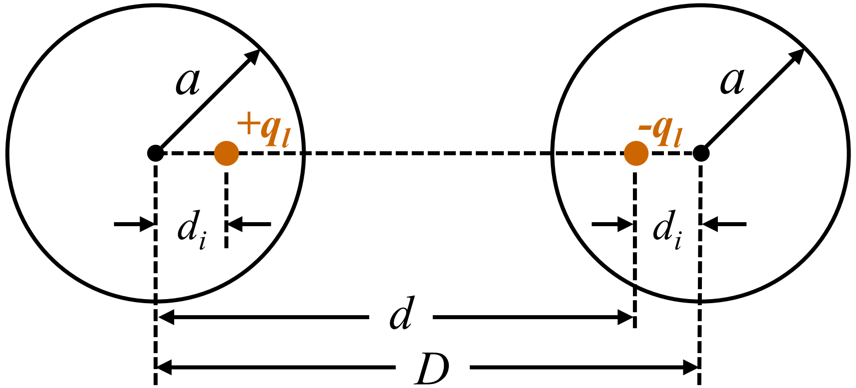


$$d_i = \frac{a^2}{d}$$

$$d = D - d_i$$

$$d = D - \frac{a^2}{d}$$

$$d^2 - Dd + a^2 = 0$$



$$d^2 - Dd + a^2 = 0$$

Quadratic Formula:

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

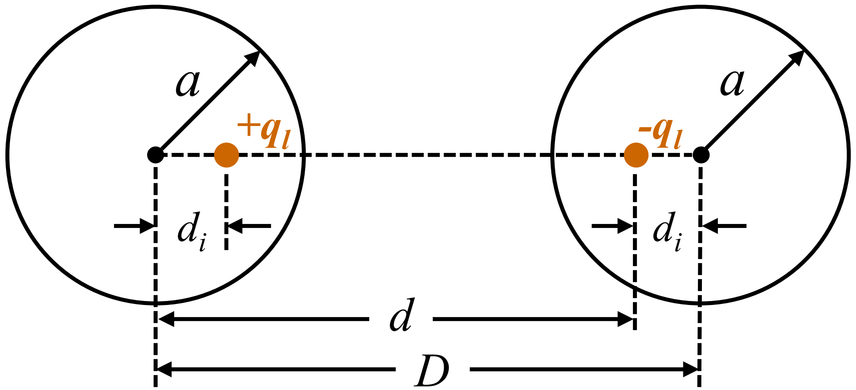
$$\downarrow$$

$$d = \frac{D + \sqrt{D^2 - 4a^2}}{2}$$

$$\frac{d}{a} = \left(\frac{D}{2a}\right) + \frac{\sqrt{D^2 - (2a)^2}}{2a}$$

$$\frac{d}{a} = \left(\frac{D}{2a}\right) + \sqrt{\left(\frac{D}{2a}\right)^2 - 1}$$

Parallel Wires (cont.)



$$V_{TOT} = \frac{q_l}{\pi\varepsilon} \ln\left(\frac{d}{a}\right)$$

$$\frac{d}{a} = \left(\frac{D}{2a}\right) + \sqrt{\left(\frac{D}{2a}\right)^2 - 1}$$

$$V_{TOT} = \frac{q_l}{\pi\varepsilon} \ln\left[\left(\frac{D}{2a}\right) + \sqrt{\left(\frac{D}{2a}\right)^2 - 1}\right]$$

$$C_l = \frac{q_l}{V_{TOT}}$$

Capacitance per unit length for parallel wires

$$\boxed{C_l = \frac{\pi\varepsilon}{\ln\left[\left(\frac{D}{2a}\right) + \sqrt{\left(\frac{D}{2a}\right)^2 - 1}\right]}}$$

Parallel Wires (cont.)

$$C_l = \frac{\pi \varepsilon}{\ln\left[\left(\frac{D}{2a}\right) + \sqrt{\left(\frac{D}{2a}\right)^2 - 1}\right]}$$

$$\varepsilon = \varepsilon_r \varepsilon_0$$